

# Variation formulas for the complex components of the Bakry-Emery-Ricci endomorphism

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## Abstract

We compute first variation formulas for the complex components of the Bakry-Emery-Ricci endomorphism along Kähler structures. Our formulas show that the principal parts of the variations are quite standard complex differential operators with particular symmetry properties on the complex decomposition of the variation of the Kähler metric. We show as application that the Soliton-Kähler-Ricci flow generated by the Soliton-Ricci flow represents a complex strictly parabolic system of the complex components of the variation of the Kähler metric.

## 1 Introduction

Let  $(X, J, g)$  be a compact Kähler manifold and  $\Omega > 0$  be a smooth volume form. It is a quite basic fact that the  $J$ -invariant part of the  $\Omega$ -Bakry-Emery-Ricci tensor is the symmetric form associated to the Chern curvature of the metric induced by  $\Omega$  over the anti-canonical bundle. We call this differential form the  $\Omega$ -Ricci form.

Moreover the  $J$ -anti-invariant part of the  $\Omega$ -Bakry-Emery-Ricci tensor measures the default of the gradient of  $\log \frac{dV_g}{\Omega}$  to be holomorphic.

In this paper we compute first variation formulas for the complex components of the endomorphism associated to the  $\Omega$ -Bakry-Emery-Ricci tensor along natural variations of Kähler structures considered in [Pal1]. Clearly these complex components correspond to the endomorphisms associated to the  $J$ -invariant and  $J$ -anti-invariant parts of the  $\Omega$ -Bakry-Emery-Ricci tensor.

Our formulas show that the principal parts of the variations are quite standard complex differential operators with particular symmetry properties on the complex decomposition of the variation of the Kähler metric. (See theorem 1.)

Our formulas will be useful for the study of the pre-scattering problem in the Fano case (see [Pal3]). We do not expect that the scattering problem in [Pal2] can be solved over general compact manifolds.

In this paper we focus also in the particular case of variations of Kähler structures with 3-symmetric covariant derivative of the variation of the metric.

(See formulas (6.7) and (6.6).) We provide a relatively simple proof of the variation formulas in this case which is independent of the general ones.

The particular variation formulas allow to show a remarkable fact. Namely that the Soliton-Kähler-Ricci flow in [Pal3] generated by the Soliton-Ricci flow in [Pal2] represents a complex strictly parabolic system of the complex components of the variation of the Kähler metric. (See the evolution equations (6.9) and (6.10).) This result can be very usefull for proving uniform estimates along the Soliton-Kähler-Ricci flow. It is known from [Pal2] that the Soliton-Ricci flow represents a strictly parabolic equation.

We explain now the main steps of the proof of the general variation formulas for the complex components of the  $\Omega$ -Bakry-Emery-Ricci endomorphism.

We compute first the variation of the  $\Omega$ -Ricci form (See lemma 2). A crucial point for the proof of this formula is is a special choice of complex geodesic coordinates. This choice is allowed by a diagonalizing property of a  $U(n)$ -action over the space of symmetric complex matrices in [Ho-Jo].

The second step is the most difficult and technical part of the proof. It consist to show that the principal part of variation of the symmetric form associated to the  $\Omega$ -Ricci form is the operator which appears in the variation of the  $\Omega$ -Bakry-Emery-Ricci tensor in [Pal1] acting on the  $J$ -anti-invariant part of the variation of the Kähler metric. (See corollary 1.)

This step is also important because it allows to give a good expression of the variation of the  $J$ -anti-linear part of the  $\Omega$ -Bakry-Emery-Ricci endomorphism. Indeed in general this is not possible by a direct computation.

The third step consist to show some special complex Weitzenböck type formulas (see lemma 4) and some symmetry relations for standard complex operators.

The final step uses in a crucial way the properties of the variations of Kähler structures obtained in [Pal1]. These properties are also of key importance for the proof of the particular variation formulas (6.7) and (6.6).

## 2 The complex components of the Bakry-Emery-Ricci endomorphism

Let  $\Omega > 0$  be a smooth volume form over an oriented Riemannian manifold  $(X, g)$ . We define the  $\Omega$ -Bakry-Emery-Ricci tensor of  $g$  as

$$\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}.$$

We define the  $\Omega$ -divergence operator of a tensor  $\alpha$  as

$$\text{div}_g^\Omega \alpha := e^f \text{div}_g (e^{-f} \alpha) = \text{div}_g \alpha - \nabla_g f \lrcorner \alpha,$$

with  $f := \log \frac{dV_g}{\Omega}$ . With this notation the first variation of the  $\Omega$ -Bakry-Emery-Ricci tensor (see [Pal1]) is given by the formula

$$2 \frac{d}{dt} \text{Ric}_{g_t}(\Omega) = \text{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t, \quad (2.1)$$

where  $\mathcal{D}_g := \hat{\nabla}_g - 2\nabla_g$ , with  $\hat{\nabla}_g$  being the symmetrization of  $\nabla_g$  acting on symmetric 2-tensors. Explicitly

$$\hat{\nabla}_g \alpha(\xi_0, \dots, \xi_p) := \sum_{j=0}^p \nabla_g \alpha(\xi_j, \xi_0, \dots, \hat{\xi}_j, \dots, \xi_p),$$

for all  $p$ -tensors  $\alpha$ . Let now  $(X, J)$  be a complex manifold. Then the volume form  $\Omega > 0$  induces a hermitian metric  $h_\Omega$  over the canonical bundle  $K_{X,J} := \Lambda_J^{n,0} T_X^*$  given by the formula

$$h_\Omega(\alpha, \beta) := \frac{n! i^{n^2} \alpha \wedge \bar{\beta}}{\Omega}.$$

By abuse of notations we will denote by  $\Omega^{-1}$  the metric  $h_\Omega$ . The dual metric  $h_\Omega^*$  on the anti-canonical bundle  $K_{X,J}^{-1} = \Lambda_J^{n,0} T_X$  is given by the formula

$$h_\Omega^*(\xi, \eta) = (-i)^{n^2} \Omega(\xi, \bar{\eta}) / n!.$$

Abusing notations again, we denote by  $\Omega$  the dual metric  $h_\Omega^*$ . We define the  $\Omega$ -Ricci form

$$\text{Ric}_J(\Omega) := i \mathcal{C}_\Omega(K_{X,J}^{-1}) = -i \mathcal{C}_{\Omega^{-1}}(K_{X,J}),$$

where  $\mathcal{C}_h(L)$  denotes the Chern curvature of a hermitian line bundle. In particular we observe the identity  $\text{Ric}_J(\omega) = \text{Ric}_J(\omega^n)$ . We remind also that for any  $J$ -invariant Kähler metric  $g$  the associated symplectic form  $\omega := gJ$  satisfies the elementary identity

$$\text{Ric}(g) = -\text{Ric}_J(\omega)J.$$

Moreover for all twice differentiable function  $f$  hold the identity

$$\nabla_g df = - (i \partial_J \bar{\partial}_J f) J + g \bar{\partial}_{T_{X,J}} \nabla_g f. \quad (2.2)$$

We infer the decomposition identity

$$\text{Ric}_g(\Omega) = -\text{Ric}_J(\Omega)J + g \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega}. \quad (2.3)$$

Let  $v \in S_{\mathbb{R}}^2 T_X^*$ ,  $\alpha \in \Lambda_{\mathbb{R}}^2 T_X^*$ . We define  $v_g^* := g^{-1}v$  and  $\alpha_g^* := \omega^{-1}\alpha$ . With this notations the decomposition formula (2.3) implies

$$\text{Ric}_g^*(\Omega) = \text{Ric}_J^*(\Omega)_g + \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega}. \quad (2.4)$$

The sections that will follow are devoted to the study of the first variation of the two complex components in (2.3) and (2.4).

### 3 The first variation of the $\Omega$ -Ricci form

Let  $\mathcal{M} \subset C^\infty(X, S_{\mathbb{R}}^2 T_X^*)$  be the space of smooth Riemannian metrics over a compact manifold  $X$ , let  $\mathcal{J} \subset C^\infty(X, \text{End}_{\mathbb{R}}(T_X))$  be the set of smooth almost complex structures and let

$$\mathcal{KS} := \left\{ (J, g) \in \mathcal{J} \times \mathcal{M} \mid g = J^* g J, \nabla_g J = 0 \right\},$$

be the space of Kähler structures. We remind that if  $A \in \text{End}_{\mathbb{R}}(T_X)$  then its transposed  $A_g^T$  with respect to  $g$  is given by  $A_g^T = g^{-1} A^* g$ . We observe that the compatibility condition  $g = J^* g J$  is equivalent to the condition  $J_g^T = -J$ . We show now an elementary formula.

**Lemma 1.** *Let  $(g_t, J_t)_t \subset \mathcal{KS}$  be a smooth path and let  $\xi$  be a smooth vector field. Then hold the first variation identity for the  $\bar{\partial}$ -operator acting on vector fields*

$$2 \left( \frac{d}{dt} \bar{\partial}_{T_X, J_t} \right) \xi = -\xi \lrcorner J_t \nabla_{g_t} \dot{J}_t + J_t \nabla_{g_t} \xi \dot{J}_t + \dot{J}_t \nabla_{g_t} \xi J_t. \quad (3.1)$$

*Proof.* The fact that in the Kähler case the Chern connection coincides with the Levi-Civita connection implies

$$2 \bar{\partial}_{T_X, J_t} \xi = \nabla_{g_t} \xi + J_t \nabla_{g_t} \xi J_t.$$

Time deriving this identity we infer

$$2 \left( \frac{d}{dt} \bar{\partial}_{T_X, J_t} \right) \xi = \dot{\nabla}_{g_t} \xi + \dot{J}_t \nabla_{g_t} \xi J_t + J_t \dot{\nabla}_{g_t} \xi J_t + J_t \nabla_{g_t} \xi \dot{J}_t. \quad (3.2)$$

On the other hand time deriving the Kähler condition  $\nabla_{g_t} J_t = 0$  we get the identity

$$\dot{\nabla}_{g_t} J_t + \nabla_{g_t} \dot{J}_t = 0,$$

which writes explicitly as

$$\dot{\nabla}_{g_t}(\xi, J_t \eta) - J_t \dot{\nabla}_{g_t}(\xi, \eta) + \nabla_{g_t} \dot{J}_t(\xi, \eta) = 0. \quad (3.3)$$

We remind now that the tensor  $\dot{\nabla}_{g_t}$  is symmetric (see [Bes] or the identity (4) in [Pal1]). Thus the identity (3.3) rewrites as

$$\dot{\nabla}_{g_t}(J_t \eta, \xi) - J_t \dot{\nabla}_{g_t}(\eta, \xi) + \nabla_{g_t} \dot{J}_t(\xi, \eta) = 0,$$

which multiplied by  $J_t$ , is equivalent to the identity

$$\dot{\nabla}_{g_t}(\eta, \xi) + J_t \dot{\nabla}_{g_t}(J_t \eta, \xi) + J_t \nabla_{g_t} \dot{J}_t(\xi, \eta) = 0,$$

i.e to the identity

$$\dot{\nabla}_{g_t} \xi + J_t \dot{\nabla}_{g_t} \xi J_t = -\xi \lrcorner J_t \nabla_{g_t} \dot{J}_t.$$

Plunging this in to the identity (3.2) we infer the variation formula (3.1).  $\square$

We introduce now a few useful notations. For any section  $S \in C^\infty(X, \text{End}_{\mathbb{R}}(T_X))$  and any  $\xi, \eta \in T_X$  we define the complex operators

$$\begin{aligned}\nabla_{g,J}^{1,0} S(\xi, \eta) &:= \frac{1}{2} \left[ \nabla_g S(\xi, \eta) - J \nabla_g S(J\xi, \eta) \right], \\ \nabla_{g,J}^{0,1} S(\xi, \eta) &:= \frac{1}{2} \left[ \nabla_g S(\xi, \eta) + J \nabla_g S(J\xi, \eta) \right].\end{aligned}$$

We define also the  $J$ -anti-linear operator

$$\nabla_{g,J}^{0,1} S \cdot \eta := \nabla_{g,J}^{0,1} S(\cdot, \eta).$$

We show now the following first variation formula for the  $\Omega$ -Ricci form.

**Lemma 2.** *Let  $(g_t, J_t)_t \subset \mathcal{KS}$  be a smooth path such that  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  and let  $\Omega > 0$  be a smooth volume form over  $X$ . Then hold the first variation formula*

$$2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega) = d \text{div}_{g_t}^\Omega (g_t \dot{J}_t). \quad (3.4)$$

*Proof.* We remind first a general identity. Let  $(L, \bar{\partial}_L, h)$  be a hermitian holomorphic line bundle over a complex manifold  $(X, J)$  and let  $D_{L,h} = \partial_{L,h} + \bar{\partial}_L$  be the induced Chern connection. We observe that for any local non-vanishing section  $\sigma \in C^\infty(U, L \setminus 0)$  over an open set  $U \subset X$ , hold the identity

$$\begin{aligned}\sigma^{-1} \partial_{L,h} \sigma(\eta) &= |\sigma|_h^{-2} h(\partial_{L,h} \sigma(\eta), \sigma) \\ &= |\sigma|_h^{-2} \left[ \eta_J^{1,0} \cdot |\sigma|_h^2 - h(\sigma, \bar{\partial}_L \sigma(\eta)) \right] \\ &= \eta_J^{1,0} \cdot \log |\sigma|_h^2 - \overline{\sigma^{-1} \bar{\partial}_L \sigma(\eta)},\end{aligned}$$

for all  $\eta \in T_X$ . We infer the formula

$$i \sigma^{-1} D_{L,h} \sigma(\eta) = i \eta_J^{1,0} \cdot \log |\sigma|_h^2 + 2 \Re(i \sigma^{-1} \bar{\partial}_L \sigma(\eta)).$$

In the case  $L = K_{X,J_t}^{-1}$  and  $h \equiv \Omega$  we get for all

$$\xi_t = \xi_{1,t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \in C^\infty(U, K_{X,J_t}^{-1} \setminus 0), \quad \xi_j \in C^\infty(U, T_X),$$

and all  $\eta \in C^\infty(U, T_X)$  the formula for the 1-form  $\alpha_t$

$$\begin{aligned}\alpha_t(\eta) &:= i \xi_t^{-1} D_{K_{X,J_t}^{-1}, \Omega} \xi_t(\eta) \\ &= i \eta_{J_t}^{1,0} \cdot \log \left[ \Omega(\xi_t, \bar{\xi}_t) / i^{n^2} \right] + 2 \Re \left( i \xi_t^{-1} \bar{\partial}_{K_{X,J_t}^{-1}, \Omega} \xi_t(\eta) \right).\end{aligned}$$

We remind also the local expression

$$\text{Ric}_{J_t}(\Omega) = i D_{K_{X,J_t}^{-1}, \Omega}^2 = d\alpha_t.$$

Let now  $f_t := \log \frac{dV_{g_t}}{\Omega}$ . We fix an arbitrary space time point  $(x_0, t_0)$  and we choose arbitrary  $g_{t_0}$ -geodesic and  $J_{t_0}$ -holomorphic coordinates  $(z_1, \dots, z_n)$  centered at the point  $x_0$  such that  $\xi_k = \frac{\partial}{\partial x_k}$ . A particular choice of a  $J_{t_0}(x_0)$ -complex basis  $(\xi_k(x_0))_k$  which is also  $\omega_{t_0}(x_0)$ -orthonormal (with  $\omega_{t_0} := g_{t_0} J_{t_0}$ ) will be made at the end of the proof. We expand now the time derivative  $\dot{\alpha}_t$ . Indeed at the time  $t_0$  hold the equalities

$$\begin{aligned}
\dot{\alpha}_t(\eta) &= \frac{1}{2} (\dot{J}_t \eta) \cdot \log \left[ \Omega(\xi_t, \bar{\xi}_t) / i^{n^2} \right] \\
&+ \frac{1}{2} \eta_{J_t}^{1,0} \cdot \sum_{l=1}^n \frac{\Omega \left( \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0}, \bar{\xi}_t \right)}{\Omega(\xi_t, \bar{\xi}_t)} \\
&- \frac{1}{2} \eta_{J_t}^{1,0} \cdot \sum_{l=1}^n \frac{\Omega \left( \xi_t, \xi_{1,t}^{0,1} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{0,1} \wedge \dots \wedge \xi_{n,t}^{0,1} \right)}{\Omega(\xi_t, \bar{\xi}_t)} \\
&+ 2 \Re e \left[ \frac{d}{dt} \left( i \xi_t^{-1} \bar{\partial}_{K_{X,J_t}^{-1}, \Omega} \xi_t(\eta) \right) \right] \\
&= - \frac{1}{2} d f_t \cdot \dot{J}_t \eta + \frac{1}{2} (\dot{J}_t \eta) \cdot \log \left[ dV_{g_t}(\xi_t, \bar{\xi}_t) / i^{n^2} \right] \\
&+ \frac{1}{2} \eta_{J_t}^{1,0} \cdot \left( \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right) \\
&- \frac{1}{2} \eta_{J_t}^{1,0} \cdot \left( \bar{\xi}_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{0,1} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{0,1} \wedge \dots \wedge \xi_{n,t}^{0,1} \right) \\
&+ \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge T \wedge \dots \wedge \xi_{n,t}^{1,0} \right],
\end{aligned}$$

with

$$T := \left( \nabla_{g_t} \dot{J}_t(\xi_l, \eta) - \nabla_{g_t} \xi_l(\dot{J}_t \eta) + \dot{J}_t \nabla_{g_t} \xi_l(\eta) \right)_{J_t}^{1,0},$$

thanks to the fact that  $\bar{\partial}_{T_X, J_t} \xi_l \equiv 0$  at time  $t_0$  and thanks to the variation formula (3.1). We define now the real 1-form  $\text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t$  by the formula

$$\left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t \right) (\xi) := \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t(\cdot, \xi), \quad \forall \xi \in T_X.$$

Expanding further the last expression of  $\dot{\alpha}_t(\eta)$ , simplifying and using the identity

$$\text{Tr}_{\mathbb{R}} A = 2 \Re e \left( \text{Tr}_{\mathbb{C}} A_J^{1,0} \right), \quad \forall A \in \text{End}_{\mathbb{R}}(T_X),$$

we infer the following expression

$$\begin{aligned}
\dot{\alpha}_t(\eta) &= -\frac{1}{2} d f_t \cdot \dot{J}_t \eta \\
&+ \frac{1}{2} \eta_{J_t}^{1,0} \cdot \left( \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right) \\
&- \frac{1}{2} \bar{\xi}_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{0,1} \wedge \dots \wedge \left( \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\eta, \xi_l) \right)_{J_t}^{0,1} \wedge \dots \wedge \xi_{n,t}^{0,1} \\
&+ \frac{1}{2} \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t \right) (\eta) \\
&+ \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge H \wedge \dots \wedge \xi_{n,t}^{1,0} \right],
\end{aligned}$$

with

$$H := \left( \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\xi_l, \eta) + \dot{J}_t \nabla_{g_t} \xi_l(\eta) \right)_{J_t}^{1,0}.$$

We rearrange the previous expression of  $\dot{\alpha}_t(\eta)$  by means of the symmetry

$$\nabla_{g_t, J_t}^{0,1} \dot{J}_t \in S_{\mathbb{R}}^2 T_X^* \otimes T_X, \quad (3.5)$$

(see lemma 7 in [Pal1]) in order to get at the time  $t_0$  the identities,

$$\begin{aligned}
\dot{\alpha}_t(\eta) &= \frac{1}{2} \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t - d f_t \cdot \dot{J}_t \right) (\eta) \\
&+ \frac{1}{2} \eta_{J_t}^{1,0} \cdot \left( \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right) \\
&+ \frac{1}{2} \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\eta, \xi_l) \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \\
&+ \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \nabla_{g_t} \xi_l(\eta) \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right] \\
&= \frac{1}{2} \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t - d f_t \cdot \dot{J}_t \right) (\eta) \\
&+ \eta \cdot \left( \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right) \\
&+ \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \nabla_{g_t} \xi_l(\eta) \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right].
\end{aligned}$$

We take now two vector fields  $\eta, \mu \in C^\infty(U, T_X)$  with constant coefficients with respect to the coordinates  $(z_1, \dots, z_n)$ . Then at the space time point  $(x_0, t_0)$  hold the identity

$$\begin{aligned}
& 2 \left( \frac{d}{dt} \text{Ric}_{J_t}(\Omega) \right) (\eta, \mu) \\
&= \left[ d \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t - d f_t \cdot \dot{J}_t \right) \right] (\eta, \mu) \\
&+ 2 \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \nabla_{g_t, \eta} \nabla_{g_t, \mu} \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right] \\
&- 2 \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \nabla_{g_t, \mu} \nabla_{g_t, \eta} \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right] \\
&= \left[ d \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t - d f_t \cdot \dot{J}_t \right) \right] (\eta, \mu) \\
&+ 2 \Re e \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \mathcal{R}_{g_t}(\eta, \mu) \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right].
\end{aligned}$$

For notation simplicity we define

$$\zeta_k := \xi_{k,t_0}^{1,0}(x_0) = \frac{\partial}{\partial z_k|_{x_0}}.$$

With this notation hold the local expression

$$\dot{J}_{t_0}(x_0) = C_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + \bar{C}_{k,\bar{l}} \zeta_l^* \otimes \bar{\zeta}_k.$$

A change of  $J_{t_0}(x_0)$ -complex basis  $(\xi_k(x_0))_k$  which is also  $\omega_{t_0}(x_0)$ -orthonormal will change the symmetric complex matrix  $C = (C_{k,\bar{l}})$  in to a matrix of type  $D := U^T C U$ , with  $U \in U(n)$ .

A standard linear algebra result (see corollary 4.4.4, pp 204-205 in Ho-Jo) shows that under this action of the unitary group the matrix  $D$  can be reduced to a real and diagonal one. Therefore we can choose the  $\omega_{t_0}(x_0)$ -orthonormal  $J_{t_0}(x_0)$ -complex basis  $(\xi_k(x_0))_k$  such that

$$\dot{J}_{t_0}(x_0) = C_l \bar{\zeta}_l^* \otimes \zeta_l + \bar{C}_l \zeta_l^* \otimes \bar{\zeta}_l,$$

with  $C_l \in \mathbb{R}$ . We write  $\eta = \eta_k \zeta_k + \bar{\eta}_k \bar{\zeta}_k$  and  $\mu = \mu_k \zeta_k + \bar{\mu}_k \bar{\zeta}_k$  at the point  $x_0$ . Moreover let  $\varphi$  be a real valued smooth function in a neighborhood of  $x_0$  such that

$$\omega_{t_0} = \frac{i}{2} \partial_{j_0} \bar{\partial}_{j_0} \varphi.$$

Then at the space time point  $(x_0, t_0)$  hold the local expression

$$\mathcal{R}_{g_t}(\eta, \mu) = - \varphi_{j,\bar{k},l,\bar{h}} (\eta_j \bar{\mu}_k - \mu_j \bar{\eta}_k) \zeta_l^* \otimes \zeta_h + \text{Conjugate}.$$



We infer the identity

$$\begin{aligned}
& 2 \Re \left[ \xi_t^{-1} \cdot \sum_{l=1}^n \xi_{1,t}^{1,0} \wedge \dots \wedge \left( \dot{J}_t \mathcal{R}_{g_t}(\eta, \mu) \xi_l \right)_{J_t}^{1,0} \wedge \dots \wedge \xi_{n,t}^{1,0} \right] \\
&= 2 \Re \left[ C_l \varphi_{k,\bar{j},l,\bar{l}} (\eta_k \bar{\mu}_j - \mu_k \bar{\eta}_j) \right] \\
&= 0,
\end{aligned}$$

at the space time point  $(x_0, t_0)$ . We deduce the variation identity

$$2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega) = d \left( \text{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t - \dot{J}_t^* d f_t \right), \quad (3.6)$$

where  $\dot{J}_t^* d f_t := d f_t \cdot \dot{J}_t$ . Finally using the symmetry identities  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  and  $\nabla_{g_t, \xi} \dot{J}_t = (\nabla_{g_t, \xi} \dot{J}_t)_{g_t}^T$  we obtain

$$2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega) = d \left[ \text{div}_{g_t}(g_t \dot{J}_t) - \nabla_{g_t} f_t \lrcorner (g_t \dot{J}_t) \right],$$

which shows the required conclusion.  $\square$

For any  $h \in C^\infty(X, (T_X^*)^{\otimes 2})$  we define respectively its  $J$ -invariant part and its  $J$ -anti-invariant part as

$$h'_J = \frac{1}{2} (h + J^* h J),$$

$$h''_J = \frac{1}{2} (h - J^* h J),$$

and we observe the trivial identities  $h'_J = g(h_g^*)_{J_t}^{1,0}$ ,  $h''_J = g(h_g^*)_{J_t}^{0,1}$ , where as usual  $h_g^* := g^{-1}h$ .

With this notations hold the following lemma.

**Corollary 1.** *Under the the assumptions of lemma 2 hold the identity*

$$\begin{aligned}
2 \frac{d}{dt} (\text{Ric}_{J_t}(\Omega) J_t) &= - \text{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'' \\
&- g_t \left( J_t \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{J}_t + \dot{J}_t \nabla_{g_t}^2 f_t J_t - J_t \nabla_{g_t}^2 f_t \dot{J}_t \right).
\end{aligned}$$

where  $\dot{g}_t'' := (\dot{g}_t)_{J_t}''$  and  $f_t := \log \frac{dV_{g_t}}{\Omega}$ .

*Proof.* We consider first the elementary decomposition

$$\text{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'' = \text{div}_{g_t} \mathcal{D}_{g_t} \dot{g}_t'' - \nabla_{g_t} f_t \lrcorner \mathcal{D}_{g_t} \dot{g}_t'',$$

and we will show the identity

$$\operatorname{div}_{g_t} \mathcal{D}_{g_t} \dot{g}_t'' = - \left( d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t \right) J_t - 2 \operatorname{Ric}_{J_t}(\omega_t) \dot{J}_t. \quad (3.7)$$

In a second time we will show the identity

$$\begin{aligned} - \nabla_{g_t} f_t \lrcorner \mathcal{D}_{g_t} \dot{g}_t &= \left( d \dot{J}_t^* d f_t \right) J_t - 2 \left( i \partial_{J_t} \bar{\partial}_{J_t} f_t \right) \dot{J}_t \\ &- g_t \left( J_t \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{J}_t + \dot{J}_t \nabla_{g_t}^2 f_t J_t - J_t \nabla_{g_t}^2 f_t \dot{J}_t \right). \end{aligned} \quad (3.8)$$

Then the conclusion will follow from the variation formula (3.6) obtained in the proof of lemma 2. We divide now the proof in a few steps.

(A) In this step we explicit in local coordinates the r.h.s of the identity (3.6). We will keep the notations for the geodesic coordinates used at the end of the proof of lemma 2 and we will denote by "Conj" the conjugate of all terms before this symbol. We consider now the local expression

$$\begin{aligned} \nabla_{g_t} \dot{J}_t &= d C_{k,\bar{l}} \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ C_{k,\bar{l}} \left( \nabla_{g_t} \bar{\zeta}_l^* \otimes \zeta_k + \bar{\zeta}_l^* \otimes \nabla_{g_t} \zeta_k \right) + \operatorname{Conj} \\ &= \left( \partial_p C_{k,\bar{l}} + A_{k,r}^p C_{r,\bar{l}} \right) \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \left( \partial_{\bar{p}} C_{k,\bar{l}} - \overline{A_{r,l}^p} C_{k,\bar{r}} \right) \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \operatorname{Conj}, \end{aligned}$$

obtained by using the expressions (7.5) and (7.6) of the complexified Levi-Civita connection in the appendix. Thus

$$\operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t = \left( \partial_r C_{r,\bar{l}} + A_{p,r}^p C_{r,\bar{l}} \right) \bar{\zeta}_l^* + \operatorname{Conj},$$

and

$$\begin{aligned} d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t &= d \left( \partial_r C_{r,\bar{l}} + A_{p,r}^p C_{r,\bar{l}} \right) \wedge \bar{\zeta}_l^* + \operatorname{Conj} \\ &= \partial_{k,r}^2 C_{r,\bar{l}} \zeta_k^* \wedge \bar{\zeta}_l^* + \partial_{k,r}^2 C_{r,\bar{l}} \bar{\zeta}_k^* \wedge \bar{\zeta}_l^* \\ &+ d \left( A_{p,r}^p C_{r,\bar{l}} \right) \wedge \bar{\zeta}_l^* + \operatorname{Conj} \\ &= \partial_{k,r}^2 C_{r,\bar{l}} \zeta_k^* \wedge \bar{\zeta}_l^* + d \left[ (\partial_p \omega_{r,\bar{s}}) \omega^{s,\bar{p}} C_{r,\bar{l}} \right] \wedge \bar{\zeta}_l^* + \operatorname{Conj}, \end{aligned}$$

since the term

$$\partial_{k,r}^2 C_{r,\bar{l}} = \partial_{r,\bar{k}}^2 C_{r,\bar{l}} = \partial_{r,\bar{l}}^2 C_{r,\bar{k}},$$

is symmetric in the indices  $k$  and  $l$ . The last equality follows from the local identity

$$\partial_{\bar{k}} C_{r,\bar{l}} = \partial_{\bar{l}} C_{r,\bar{k}} , \quad (3.9)$$

which is equivalent to the identity

$$\bar{\partial}_{T_X, J_t} \dot{J}_t \equiv 0 .$$

(See lemma 7 in [Pal1].) Thus at the point  $p_0$  where the geodesic coordinates are centered hold the expression

$$\begin{aligned} d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t &= \partial_{k,r}^2 C_{r,\bar{l}} \zeta_k^* \wedge \bar{\zeta}_l^* \\ &+ \left( \partial_{p,\bar{k}}^2 \omega_{r,\bar{p}} C_{r,\bar{l}} \right) \bar{\zeta}_k^* \wedge \bar{\zeta}_l^* + \text{Conj} , \end{aligned}$$

i.e.

$$\begin{aligned} d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t &= \left( \partial_{k,r}^2 C_{r,\bar{l}} - \partial_{l,\bar{r}}^2 \overline{C_{r,\bar{k}}} \right) \zeta_k^* \wedge \bar{\zeta}_l^* \\ &- R_{k,\bar{l}} C_{l,\bar{l}} \zeta_k^* \wedge \zeta_l^* - R_{l,\bar{k}} C_{l,\bar{l}} \bar{\zeta}_k^* \wedge \bar{\zeta}_l^* . \end{aligned} \quad (3.10)$$

**(B)** In this part we prove the identity (3.7). We keep the previous notations for the geodesic coordinates and we consider arbitrary vector fields  $\xi, \eta$  with constant coefficients with respect to this coordinates. Using the local expression (3.10) we infer the identity

$$\begin{aligned} d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t (J_t \xi, \eta) &= i \left( \partial_{k,r}^2 C_{r,\bar{l}} - \partial_{l,\bar{r}}^2 \overline{C_{r,\bar{k}}} \right) (\xi_k \bar{\eta}_l + \bar{\xi}_l \eta_k) \\ &+ i R_{k,\bar{l}} C_{l,\bar{l}} (\xi_l \eta_k - \xi_k \eta_l) \\ &+ i R_{l,\bar{k}} C_{l,\bar{l}} (\bar{\xi}_k \bar{\eta}_l - \bar{\xi}_l \bar{\eta}_k) , \end{aligned}$$

at the point  $p_0$ . On the other hand the general identity

$$2(\dot{g}_t^*)_{J_t}^{0,1} = -J_t \dot{J}_t - (J_t \dot{J}_t)_{g_t}^T ,$$

(see lemma 3, identity 15 in [Pal1]) combined with the symmetry assumption  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  implies

$$(\dot{g}_t'')^* = (\dot{g}_t^*)_{J_t}^{0,1} = -J_t \dot{J}_t .$$

Using this last identity we expand the therm

$$\operatorname{div}_{g_t} \mathcal{D}_{g_t} \dot{g}_t''(\xi, \eta) = 2 \nabla_{g_t} \mathcal{D}_{g_t} \dot{g}_t''(\zeta_l, \bar{\zeta}_l, \xi, \eta) + 2 \nabla_{g_t} \mathcal{D}_{g_t} \dot{g}_t''(\bar{\zeta}_l, \zeta_l, \xi, \eta) .$$

We obtain the equalities

$$\begin{aligned}
2 \nabla_{g_t} \mathcal{D}_{g_t} \dot{g}_t'' (\zeta_l, \bar{\zeta}_l, \xi, \eta) &= - 2 \nabla_{g_t}^2 \dot{g}_t'' (\zeta_l, \bar{\zeta}_l, \xi, \eta) + 2 \nabla_{g_t}^2 \dot{g}_t'' (\zeta_l, \xi, \bar{\zeta}_l, \eta) \\
&+ 2 \nabla_{g_t}^2 \dot{g}_t'' (\zeta_l, \eta, \bar{\zeta}_l, \xi) \\
&= - 2g \left( J_t \nabla_{g_t}^2 \dot{J}_t (\zeta_l, \xi, \bar{\zeta}_l), \eta \right) \\
&- 2g \left( J_t \nabla_{g_t}^2 \dot{J}_t (\zeta_l, \eta, \bar{\zeta}_l), \xi \right) \\
&+ 2g \left( J_t \nabla_{g_t}^2 \dot{J}_t (\zeta_l, \bar{\zeta}_l, \xi), \eta \right) .
\end{aligned}$$

We compute now the local expression of the tensor  $\nabla_{g_t}^2 \dot{J}_t$ . Taking the covariant derivative of the local expression of the tensor  $\nabla_{g_t} \dot{J}_t$ , obtained in the beginning of step **(A)**, we infer the expression at the point  $p_0$

$$\begin{aligned}
\nabla_{g_t}^2 \dot{J}_t &= \left( d \partial_p C_{k, \bar{l}} + d A_{k, h}^p C_{h, \bar{l}} \right) \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\
&+ \left( d \partial_{\bar{p}} C_{k, \bar{l}} - d \overline{A_{h, l}^p} C_{k, \bar{h}} \right) \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \text{Conj} \\
&= \partial_{r, p}^2 C_{k, \bar{l}} \zeta_r^* \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\
&+ \left( \partial_{\bar{r}, p}^2 C_{k, \bar{l}} + C_{l, \bar{l}} \varphi_{p, \bar{r}, l, \bar{k}} \right) \bar{\zeta}_r^* \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\
&+ \left( \partial_{r, \bar{p}}^2 C_{k, \bar{l}} - C_{k, \bar{k}} \varphi_{r, \bar{p}, k, \bar{l}} \right) \zeta_r^* \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\
&+ \partial_{\bar{r}, \bar{p}}^2 C_{k, \bar{l}} \bar{\zeta}_r^* \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \text{Conj} .
\end{aligned}$$

Using this last expression we expand the terms (i.e the sums, by the Einstein convention)

$$\begin{aligned}
J_t \nabla_{g_t}^2 \dot{J}_t (\zeta_l, \xi, \bar{\zeta}_l) &= i \left( \partial_{l, p}^2 C_{k, \bar{l}} + C_{l, \bar{l}} \varphi_{p, \bar{l}, l, \bar{k}} \right) \xi_p \zeta_k \\
&+ i \left( \partial_{l, \bar{p}}^2 C_{k, \bar{l}} + C_{k, \bar{k}} R_{k, \bar{p}} \right) \bar{\xi}_p \zeta_k, \\
J_t \nabla_{g_t}^2 \dot{J}_t (\zeta_l, \bar{\zeta}_l, \xi) &= i \left( \partial_{l, \bar{l}}^2 C_{k, \bar{p}} + C_{k, \bar{k}} R_{k, \bar{p}} \right) \bar{\xi}_p \zeta_k \\
&- i \left( \partial_{l, \bar{l}}^2 \overline{C_{k, \bar{p}}} - C_{p, \bar{p}} R_{k, \bar{p}} \right) \xi_p \bar{\zeta}_k .
\end{aligned}$$

We infer the expression

$$\begin{aligned}
& 2\nabla_{g_t} \mathcal{D}_{g_t} \dot{g}_t'' (\zeta_l, \bar{\zeta}_l, \xi, \eta) \\
&= -i \left[ (\partial_{l,p}^2 C_{k,\bar{l}} + C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}}) \xi_p + (\partial_{l,\bar{p}}^2 C_{k,\bar{l}} + C_{k,\bar{k}} R_{k,\bar{p}}) \bar{\xi}_p \right] \bar{\eta}_k \\
&- i \left[ (\partial_{l,p}^2 C_{k,\bar{l}} + C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}}) \eta_p + (\partial_{l,\bar{p}}^2 C_{k,\bar{l}} + C_{k,\bar{k}} R_{k,\bar{p}}) \bar{\eta}_p \right] \bar{\xi}_k \\
&+ i \left( \partial_{l,\bar{l}}^2 C_{k,\bar{p}} + C_{k,\bar{k}} R_{k,\bar{p}} \right) \bar{\xi}_p \bar{\eta}_k - i \left( \partial_{l,\bar{l}}^2 \overline{C_{k,\bar{p}}} - C_{p,\bar{p}} R_{k,\bar{p}} \right) \xi_p \eta_k \\
&= -i \left( \partial_{l,p}^2 C_{k,\bar{l}} + C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}} \right) \xi_p \bar{\eta}_k \\
&- i \left[ (\partial_{l,p}^2 C_{k,\bar{l}} + C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}}) \eta_p + (\partial_{l,\bar{p}}^2 C_{k,\bar{p}} + C_{k,\bar{k}} R_{k,\bar{p}}) \bar{\eta}_p \right] \bar{\xi}_k \\
&- i \left( \partial_{l,\bar{l}}^2 \overline{C_{k,\bar{p}}} - C_{p,\bar{p}} R_{k,\bar{p}} \right) \xi_p \eta_k,
\end{aligned}$$

thanks to the identity (3.9). Moreover simplifying the therms

$$C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}} \xi_p \bar{\eta}_k, \quad C_{l,\bar{l}} \varphi_{p,\bar{l},l,\bar{k}} \eta_p \bar{\xi}_k,$$

with their conjugates (We remind that  $C_{l,\bar{l}} = \overline{C_{l,\bar{l}}}$  thanks to our special choice of geodesic coordinates.) and rearranging the conjugate therms we obtain

$$\begin{aligned}
\operatorname{div}_{g_t} \mathcal{D}_{g_t} \dot{g}_t'' (\xi, \eta) &= -i \partial_{l,p}^2 C_{k,\bar{l}} (\xi_p \bar{\eta}_k + \bar{\xi}_k \eta_p) \\
&- i \left( \partial_{l,\bar{l}}^2 C_{k,\bar{p}} + C_{k,\bar{k}} R_{k,\bar{p}} \right) \bar{\xi}_k \bar{\eta}_p \\
&+ i \left( \partial_{l,\bar{l}}^2 C_{k,\bar{p}} - C_{p,\bar{p}} R_{p,\bar{k}} \right) \bar{\xi}_p \bar{\eta}_k + \operatorname{Conj} \\
&= -i \partial_{l,p}^2 C_{k,\bar{l}} (\xi_p \bar{\eta}_k + \bar{\xi}_k \eta_p) \\
&+ i \left( \partial_{l,\bar{l}}^2 C_{p,\bar{k}} - \partial_{l,\bar{l}}^2 C_{k,\bar{p}} \right) \bar{\xi}_k \bar{\eta}_p \\
&- 2i C_{k,\bar{k}} R_{k,\bar{p}} \bar{\xi}_k \bar{\eta}_p + \operatorname{Conj} \\
&= -i \partial_{l,p}^2 C_{k,\bar{l}} (\xi_p \bar{\eta}_k + \bar{\xi}_k \eta_p) \\
&+ i \left( \partial_{l,\bar{l}}^2 C_{p,\bar{k}} - \partial_{l,\bar{l}}^2 C_{k,\bar{p}} \right) \bar{\xi}_k \bar{\eta}_p + \operatorname{Conj} \\
&- 2 \operatorname{Ric}_{J_t} (\omega_t) (\dot{J}_t \xi, \eta),
\end{aligned}$$

at the point  $p_0$ . We remind now that for any tensor  $\alpha$  and any smooth vector fields  $v, w$  such that  $\nabla_{g_t} w(q) = 0$  at some point  $q$  hold the well known identity  $\nabla_{v,w}^2 \alpha(q) = \nabla_v \nabla_w \alpha(q)$ . By applying this to the identity  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  we infer the formula

$$\nabla_{V,W}^2 \dot{J}_t = \left( \nabla_{V,W}^2 \dot{J}_t \right)_{g_t}^T,$$

for any smooth vector fields  $V, W$ . In particular for all indices  $r, p$ , the identity

$$\nabla_{\zeta_r, \zeta_p}^2 \dot{J}_t = \left( \nabla_{\zeta_r, \zeta_p}^2 \dot{J}_t \right)_{g_t}^T,$$

is equivalent to the local symmetry identity

$$\partial_{r,p}^2 C_{k,\bar{l}} = \partial_{r,p}^2 C_{l,\bar{k}},$$

for all indices  $r, p, k, l$ . Moreover for all indices  $l$ , the identity

$$\nabla_{\zeta_l, \bar{\zeta}_l}^2 \dot{J}_t = \left( \nabla_{\zeta_l, \bar{\zeta}_l}^2 \dot{J}_t \right)_{g_t}^T,$$

implies the symmetry identity

$$\partial_{l,\bar{l}}^2 C_{k,\bar{p}} + C_{k,\bar{k}} R_{k,\bar{p}} = \partial_{l,\bar{l}}^2 C_{p,\bar{k}} + C_{p,\bar{p}} R_{p,\bar{k}}, \quad (3.11)$$

for all indices  $k, p$ . We infer the expressions

$$\begin{aligned} \operatorname{div}_{g_t} \mathcal{D}_{g_t} \dot{g}_t''(\xi, \eta) &= -i \partial_{l,p}^2 C_{l,\bar{k}} (\xi_p \bar{\eta}_k + \bar{\xi}_k \eta_p) \\ &+ i (C_{k,\bar{k}} R_{k,\bar{p}} - C_{p,\bar{p}} R_{p,\bar{k}}) \bar{\xi}_k \bar{\eta}_p + \operatorname{Conj} \\ &- 2 \operatorname{Ric}_{J_t}(\omega_t)(\dot{J}_t \xi, \eta) \\ &= -i \partial_{p,l}^2 C_{l,\bar{k}} (\xi_p \bar{\eta}_k + \bar{\xi}_k \eta_p) \\ &+ i C_{k,\bar{k}} R_{k,\bar{p}} (\bar{\xi}_k \bar{\eta}_p - \bar{\xi}_p \bar{\eta}_k) + \operatorname{Conj} \\ &- 2 \operatorname{Ric}_{J_t}(\omega_t)(\dot{J}_t \xi, \eta) \\ &= -d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t (J_t \xi, \eta) - 2 \operatorname{Ric}_{J_t}(\omega_t)(\dot{J}_t \xi, \eta), \end{aligned}$$

thanks to the expression of the therm  $d \operatorname{Tr}_{\mathbb{R}} \nabla_{g_t} \dot{J}_t (J_t \xi, \eta)$  obtained in the beginning of step **(B)**. We deduce the required identity (3.7).

(C) We show now the formula (3.8). We decompose the therm

$$\begin{aligned}
- \mathcal{D}_{g_t} \dot{g}_t''(\nabla_{g_t} f_t, \xi, \eta) &= \nabla_{g_t} \dot{g}_t''(\nabla_{g_t} f_t, \xi, \eta) - \nabla_{g_t} \dot{g}_t''(\xi, \nabla_{g_t} f_t, \eta) \\
&- \nabla_{g_t} \dot{g}_t''(\eta, \nabla_{g_t} f_t, \xi) \\
&= - g_t \left( J_t \nabla_{g_t} \dot{J}_t(\nabla_{g_t} f_t, \xi), \eta \right) \\
&+ g_t \left( J_t \nabla_{g_t} \dot{J}_t(\xi, \nabla_{g_t} f_t), \eta \right) \\
&+ g_t \left( J_t \nabla_{g_t} \dot{J}_t(\eta, \nabla_{g_t} f_t), \xi \right) \\
&= - g_t \left( J_t \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\nabla_{g_t} f_t, \xi), \eta \right) \\
&+ g_t \left( J_t \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\xi, \nabla_{g_t} f_t), \eta \right) \\
&+ g_t \left( J_t \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\eta, \nabla_{g_t} f_t), \xi \right) \\
&+ g_t \left( J_t \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\nabla_{g_t} f_t, \eta), \xi \right),
\end{aligned}$$

thanks to the symmetry property (3.5). We observe also that the endomorphism  $A := J_t \dot{J}_t$  satisfies the symmetry identities

$$\xi \lrcorner \nabla_{g, J}^{1,0} A = \left( \xi \lrcorner \nabla_{g, J}^{1,0} A \right)_g^T, \quad (3.12)$$

$$\xi \lrcorner \nabla_{g, J}^{0,1} A = \left( \xi \lrcorner \nabla_{g, J}^{0,1} A \right)_g^T, \quad (3.13)$$

which are direct consequence of the equality  $A = A_g^T$ . We infer

$$\begin{aligned}
g_t \left( J_t \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\xi, \nabla_{g_t} f_t), \eta \right) &= g_t \left( J_t \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\xi, \eta), \nabla_{g_t} f_t \right), \\
g_t \left( J_t \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\nabla_{g_t} f_t, \eta), \xi \right) &= g_t \left( J_t \nabla_{g_t, J_t}^{0,1} \dot{J}_t(\nabla_{g_t} f_t, \xi), \eta \right).
\end{aligned}$$

Thus

$$\begin{aligned}
- \mathcal{D}_{g_t} \dot{g}_t''(\nabla_{g_t} f_t, \xi, \eta) &= df_t \cdot J_t \left( \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\xi, \eta) + \nabla_{g_t, J_t}^{1,0} \dot{J}_t(\eta, \xi) \right) \\
&- g_t \left( \nabla_{g_t} \dot{J}_t(J_t \nabla_{g_t} f_t, \xi), \eta \right).
\end{aligned}$$

We use this last identity in the decomposition of the therm

$$\begin{aligned}
\left( d J_t^* d f_t \right) (J_t \xi, \eta) &= \nabla_{g_t} d f_t (J_t \xi, \dot{J}_t \eta) + d f_t \cdot \nabla_{g_t} \dot{J}_t (J_t \xi, \eta) \\
&- \nabla_{g_t} d f_t (\eta, \dot{J}_t J_t \xi) - d f_t \cdot \nabla_{g_t} \dot{J}_t (\eta, J_t \xi) \\
&= g_t \left( \nabla_{g_t}^2 f_t J_t \xi, \dot{J}_t \eta \right) + \nabla_{g_t} d f_t (J_t \dot{J}_t \xi, \eta) \\
&+ d f_t \cdot J_t \left( \nabla_{g_t, J_t}^{1,0} \dot{J}_t (\xi, \eta) + \nabla_{g_t, J_t}^{1,0} \dot{J}_t (\eta, \xi) \right) \\
&= g_t \left( \dot{J}_t \nabla_{g_t}^2 f_t J_t \xi, \eta \right) + \nabla_{g_t} d f_t (J_t \dot{J}_t \xi, \eta) \\
&- \mathcal{D}_{g_t} \dot{g}_t'' (\nabla_{g_t} f_t, \xi, \eta) + g_t \left( \nabla_{g_t} \dot{J}_t (J_t \nabla_{g_t} f_t, \xi), \eta \right).
\end{aligned}$$

Combining this with the decomposition formula (2.2) we infer the required identity (3.8).  $\square$

Combining the time derivative of the identity (2.3) with the variation formula (2.1) and with corollary 1 we infer immediately the following corollary.

**Corollary 2.** *Under the the assumptions of lemma 2 hold the identity*

$$\begin{aligned}
2 \frac{d}{dt} \left( g_t \bar{\partial}_{\tau_{X,J}} \nabla_{g_t} f_t \right) &= \operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t' \\
&- g_t \left( J_t \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{J}_t + \dot{J}_t \nabla_{g_t}^2 f_t J_t - J_t \nabla_{g_t}^2 f_t \dot{J}_t \right),
\end{aligned}$$

where  $\dot{g}_t' := (\dot{g}_t)'_{J_t}$  and  $f_t := \log \frac{dV_{g_t}}{\Omega}$ .

## 4 Weitzenböck type formulas

We denote by  $P_g^*$  the formal adjoint of an operator  $P_g$  depending on  $g$ . We observe that the operator

$$P_g^{*\Omega} := e^f P_g^* (e^{-f} \bullet),$$

is the formal adjoint of  $P_g$  with respect to the scalar product  $\int_X \langle \cdot, \cdot \rangle_g \Omega$ . With this notation hold the identity  $\nabla_g^{*\Omega} = -\operatorname{div}_g^\Omega$ . As in [Pal2] we define the  $\Omega$ -Laplacian

$$\Delta_g^\Omega := \nabla_g^{*\Omega} \nabla_g = \Delta_g + \nabla_g f \lrcorner \nabla_g,$$

with  $f := \log \frac{dV_g}{\Omega}$  and we remind the following result obtained in [Pal2].



**Lemma 3.** For any  $g \in \mathcal{M}$  and  $u \in C^\infty(X, S^2 T_X^*)$  hold the formula

$$(\operatorname{div}_g^\Omega \mathcal{D}_g u)_g^* = \frac{1}{2} \nabla_{T_X, g}^* \nabla_{T_X, g} u_g^* + \frac{1}{2} \left( \nabla_{T_X, g}^* \nabla_{T_X, g} u_g^* \right)_g^T - \Delta_g^\Omega u_g^*,$$

where  $\nabla_{T_X, g}$  denotes the covariant exterior derivative acting on  $T_X$ -valued differential forms.

Using the expressions (7.7) and (7.8) of the adjoint of the standard complex operators in the subsection 7.3 of the appendix we can show the following complex Weitzenböck type formulas.

**Lemma 4.** Consider  $(J, g) \in \mathcal{KS}$  and let  $A \in C^\infty(X, T_{X, -J}^* \otimes_{\mathbb{C}} T_{X, J})$ ,  $B \in C^\infty(X, T_{X, J}^* \otimes_{\mathbb{C}} T_{X, J})$  be endomorphism sections. Then hold the identities

$$\Delta_g A = \partial_{T_X, J}^{*g} \partial_{T_X, J}^g A - A \operatorname{Ric}_g^* - \operatorname{Ric}_g^* A, \quad (4.1)$$

$$\Delta_g B = \bar{\partial}_{T_X, J}^{*g} \bar{\partial}_{T_X, J}^g B - B \operatorname{Ric}_g^* + \operatorname{Ric}_g^* B. \quad (4.2)$$

*Proof.* We choose  $g$ -orthonormal and  $J$ -holomorphic coordinates centered at an arbitrary point  $p_0$ . Let  $A = C_{k, \bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + \operatorname{Conj}$ , be the local expression of  $A$  and we consider the local expression

$$\begin{aligned} \nabla_g A &= d C_{k, \bar{l}} \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ C_{k, \bar{l}} (\nabla_{g_t} \bar{\zeta}_l^* \otimes \zeta_k + \bar{\zeta}_l^* \otimes \nabla_{g_t} \zeta_k) + \operatorname{Conj} \\ &= \left( \partial_p C_{k, \bar{l}} + A_{k, r}^p C_{r, \bar{l}} \right) \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \left( \partial_{\bar{p}} C_{k, \bar{l}} - \overline{A_{r, l}^p} C_{k, \bar{r}} \right) \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \operatorname{Conj}, \end{aligned}$$

obtained by using by the expression (7.5) of the complexified Levi-Civita connection in the appendix. Taking again the covariant derivative we infer the expression at the point  $p_0$

$$\begin{aligned} \nabla_g^2 A &= \left( d \partial_p C_{k, \bar{l}} + d A_{k, h}^p C_{h, \bar{l}} \right) \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \left( d \partial_{\bar{p}} C_{k, \bar{l}} - d \overline{A_{h, l}^p} C_{k, \bar{h}} \right) \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \operatorname{Conj} \\ &= \partial_{r, p}^2 C_{k, \bar{l}} \zeta_r^* \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \left( \partial_{\bar{r}, p}^2 C_{k, \bar{l}} + C_{h, \bar{l}} \varphi_{p, \bar{r}, h, \bar{k}} \right) \bar{\zeta}_r^* \otimes \zeta_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \left( \partial_{r, \bar{p}}^2 C_{k, \bar{l}} - C_{k, \bar{h}} \varphi_{r, \bar{p}, h, \bar{l}} \right) \zeta_r^* \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k \\ &+ \partial_{\bar{r}, \bar{p}}^2 C_{k, \bar{l}} \bar{\zeta}_r^* \otimes \bar{\zeta}_p^* \otimes \bar{\zeta}_l^* \otimes \zeta_k + \operatorname{Conj}. \end{aligned}$$

Thus

$$\begin{aligned}\Delta_g A &= -2 \nabla_{g, \zeta_k} \nabla_{g, \bar{\zeta}_k} A - 2 \nabla_{g, \bar{\zeta}_k} \nabla_{g, \zeta_k} A \\ &= -2 \left( 2 \partial_{r, \bar{r}}^2 C_{k, \bar{l}} + C_{k, \bar{h}} R_{h, \bar{l}} - C_{h, \bar{l}} R_{h, \bar{k}} \right) \bar{\zeta}_l^* \otimes \zeta_k + \text{Conj},\end{aligned}$$

at the point  $p_0$ . Moreover using the local expression

$$\partial_{T_{X, J}}^g \zeta_k = A_{l, k}^p \zeta_p^* \otimes \zeta_l + \text{Conj},$$

we obtain

$$\partial_{T_{X, J}}^g A = \left( \partial_p C_{k, \bar{l}} + A_{k, h}^p C_{h, \bar{l}} \right) (\zeta_p^* \wedge \bar{\zeta}_l^*) \otimes \zeta_k + \text{Conj},$$

and

$$\nabla_{g, J}^{0,1} \partial_{T_{X, J}}^g A = \left( \partial_{p, \bar{r}}^2 C_{k, \bar{l}} + \varphi_{p, \bar{r}, h, \bar{k}} C_{h, \bar{l}} \right) \bar{\zeta}_r^* \otimes (\zeta_p^* \wedge \bar{\zeta}_l^*) \otimes \zeta_k + \text{Conj},$$

Thus using the expression (7.7) we expand the therm

$$\begin{aligned}\partial_{T_{X, J}}^{*g} \partial_{T_{X, J}}^g A &= -2 \text{Tr}_g \left( \nabla_{g, J}^{0,1} \partial_{T_{X, J}}^g A \right) \\ &= -4 \nabla_{g, J}^{0,1} \partial_{T_{X, J}}^g A (\bar{\zeta}_r, \zeta_r, \cdot) \\ &= -4 \left( \partial_{r, \bar{r}}^2 C_{k, \bar{l}} - C_{h, \bar{l}} R_{h, \bar{k}} \right) \bar{\zeta}_l^* \otimes \zeta_k + \text{Conj}.\end{aligned}$$

We infer the required formula for  $A$ . Let now  $B = B_{k, \bar{l}} \zeta_k^* \otimes \zeta_l + \text{Conj}$ , be the local expression of  $B$ . Expanding the identity

$$\nabla_g B = d B_{k, \bar{l}} \otimes \zeta_k^* \otimes \zeta_l + B_{k, \bar{l}} \nabla_g \zeta_k^* \otimes \zeta_l + B_{k, \bar{l}} \zeta_k^* \otimes \nabla_g \zeta_l + \text{Conj},$$

we infer

$$\begin{aligned}\nabla_g B &= \left( \partial_p B_{k, \bar{l}} + B_{k, \bar{h}} A_{l, h}^p - B_{h, \bar{l}} A_{h, k}^p \right) \zeta_p^* \otimes \zeta_k^* \otimes \zeta_l \\ &\quad + \partial_{\bar{p}} B_{k, \bar{l}} \bar{\zeta}_p^* \otimes \zeta_k^* \otimes \zeta_l + \text{Conj},\end{aligned}$$

at the point  $p_0$ . We deduce the local expression

$$\begin{aligned}\nabla_g^2 B &= \partial_{r, p}^2 B_{k, \bar{l}} \zeta_r^* \otimes \zeta_p^* \otimes \zeta_k^* \otimes \zeta_l \\ &\quad + \left( \partial_{p, \bar{r}}^2 B_{k, \bar{l}} + B_{k, \bar{h}} \varphi_{p, \bar{r}, h, \bar{l}} - B_{h, \bar{l}} \varphi_{p, \bar{r}, k, \bar{h}} \right) \bar{\zeta}_r^* \otimes \zeta_p^* \otimes \zeta_k^* \otimes \zeta_l \\ &\quad + \partial_{r, \bar{p}}^2 B_{k, \bar{l}} \zeta_r^* \otimes \bar{\zeta}_p^* \otimes \zeta_k^* \otimes \zeta_l + \partial_{\bar{r}, \bar{p}}^2 B_{k, \bar{l}} \bar{\zeta}_r^* \otimes \bar{\zeta}_p^* \otimes \zeta_k^* \otimes \zeta_l + \text{Conj}.\end{aligned}$$

Thus

$$\begin{aligned}\Delta_g B &= -2 \nabla_{g, \zeta_k} \nabla_{g, \bar{\zeta}_k} B - 2 \nabla_{g, \bar{\zeta}_k} \nabla_{g, \zeta_k} B \\ &= -2 \left( 2 \partial_{r, \bar{r}}^2 B_{k, \bar{l}} - B_{k, \bar{h}} R_{h, \bar{l}} + B_{h, \bar{l}} R_{k, \bar{h}} \right) \zeta_k^* \otimes \zeta_l + \text{Conj} ,\end{aligned}$$

at the point  $p_0$ . On the other hand deriving the local expression

$$\bar{\partial}_{T_{X,J}} B = - \partial_{\bar{p}} B_{k, \bar{l}} (\zeta_k^* \wedge \bar{\zeta}_p^*) \otimes \zeta_l + \text{Conj} ,$$

we infer

$$\nabla_{g,J}^{1,0} \bar{\partial}_{T_{X,J}} B = - \partial_{r, \bar{p}}^2 B_{k, \bar{l}} \zeta_r^* \otimes (\zeta_k^* \wedge \bar{\zeta}_p^*) \otimes \zeta_l + \text{Conj} ,$$

and thus using the expression (7.8) we obtain

$$\begin{aligned}\bar{\partial}_{T_{X,J}}^* \bar{\partial}_{T_{X,J}} B &= -2 \text{Tr}_g \left( \nabla_{g,J}^{1,0} \bar{\partial}_{T_{X,J}} B \right) \\ &= -4 \nabla_{g,J}^{1,0} \bar{\partial}_{T_{X,J}} B (\zeta_r, \bar{\zeta}_r, \cdot) \\ &= -4 \partial_{r, \bar{r}}^2 B_{k, \bar{l}} \zeta_k^* \otimes \zeta_l + \text{Conj} ,\end{aligned}$$

at the point  $p_0$ . The conclusion follows from the local expression of  $\Delta_g B$ .  $\square$

We observe that with our conventions our adjoint operators

$$\nabla_{T_{X,g}}^* , \quad \partial_{T_{X,J}}^{*g} , \quad \bar{\partial}_{T_{X,J}}^{*g} ,$$

differ by the ones usually defined in the literature by a degree multiplicative factor. This is due to the fact that with our conventions the metric induced on the space of forms is the restriction of the metric on the space of tensors (without degree multiplicative factors). We refer to the appendix in [Pal1] for more details. We observe also that the formal adjoint of the  $\partial_{T_{X,J}}^g$ -operator with respect to the hermitian product  $\int_X \langle \cdot, \cdot \rangle_{gJ} \Omega$ , is the operator

$$\partial_{T_{X,J}}^{*g, \Omega} := e^f \partial_{T_{X,J}}^{*g} (e^{-f} \bullet) .$$

In a similar way the formal adjoint of the  $\bar{\partial}_{T_{X,J}}$ -operator with respect to the hermitian product  $\int_X \langle \cdot, \cdot \rangle_{gJ} \Omega$ , is the operator

$$\bar{\partial}_{T_{X,J}}^{*g, \Omega} := e^f \bar{\partial}_{T_{X,J}}^{*g} (e^{-f} \bullet) .$$

With this notations hold the following lemma.

**Lemma 5.** Consider  $(J, g) \in \mathcal{KS}$  and let  $A \in C^\infty(X, T_{X,-J}^* \otimes_{\mathbb{C}} T_{X,J}), B \in C^\infty(X, T_{X,J}^* \otimes_{\mathbb{C}} T_{X,J})$  be  $g$ -symmetric endomorphism sections. Then hold the symmetry identities

$$\partial_{T_{X,J}}^{*g,\Omega} \partial_{T_{X,J}}^g A = \left( \partial_{T_{X,J}}^{*g,\Omega} \partial_{T_{X,J}}^g A \right)_g^T, \quad (4.3)$$

$$\partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g,\Omega} A + \frac{1}{2} \bar{\partial}_{T_{X,J}}^{*g,\Omega} \partial_{T_{X,J}}^g A = A \bar{\partial}_{T_{X,J}} \nabla_g f, \quad (4.4)$$

$$\begin{aligned} \bar{\partial}_{T_{X,J}}^{*g,\Omega} \bar{\partial}_{T_{X,J}} B &= \left( \bar{\partial}_{T_{X,J}}^{*g,\Omega} \bar{\partial}_{T_{X,J}} B \right)_g^T \\ &- 2 \nabla_g f \lrcorner \left( \nabla_{g,J}^{1,0} B - \nabla_{g,J}^{0,1} B \right) \\ &+ 2 [\text{Ric}_g^*, B], \end{aligned} \quad (4.5)$$

$$\bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^{*g,\Omega} B + \frac{1}{2} \partial_{T_{X,J}}^{*g,\Omega} \bar{\partial}_{T_{X,J}} B = B \bar{\partial}_{T_{X,J}} \nabla_g f. \quad (4.6)$$

*Proof.* Using the definition of the adjoint operator  $\partial_{T_{X,J}}^{*g,\Omega}$  and the expression (7.7) we expand the term

$$\begin{aligned} \partial_{T_{X,J}}^{*g,\Omega} \partial_{T_{X,J}}^g A &= -2e^f \text{Tr}_g \left[ \nabla_{g,J}^{0,1} \left( e^{-f} \partial_{T_{X,J}}^g A \right) \right] \\ &= \partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A + 2 \text{Tr}_g \left( \bar{\partial}_J f \otimes_J \partial_{T_{X,J}}^g A \right) \\ &= \partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A + 2 \nabla_g f \lrcorner \nabla_{g,J}^{1,0} A. \end{aligned}$$

Moreover the identity (4.1) in lemma 4 implies the equality

$$\partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A = \left( \partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A \right)_g^T.$$

This combined with the identity (3.12) implies the required identity (4.3). Using the definition of the operator  $\bar{\partial}_{T_{X,J}}^{*g,\Omega}$  and the expression (7.8) we infer

$$\begin{aligned} \bar{\partial}_{T_{X,J}}^{*g,\Omega} \partial_{T_{X,J}}^g A &= -2e^f \text{Tr}_g \left[ \nabla_{g,J}^{1,0} \left( e^{-f} \partial_{T_{X,J}}^g A \right) \right] \\ &= \bar{\partial}_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A + 2 \text{Tr}_g \left( \partial_J f \otimes_J \partial_{T_{X,J}}^g A \right) \\ &= \bar{\partial}_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g A - 2 \nabla_{g,J}^{1,0} A \nabla_g f. \end{aligned}$$

We observe in fact that at the center  $p_0$  of any  $J$ -holomorphic geodesic coordi-

nates hold the following equalities

$$\begin{aligned}
\mathrm{Tr}_g \left( \partial_J f \otimes_J \partial_{T_{X,J}}^g A \right) &= 2 f_l \left( \bar{\zeta}_l \lrcorner_J \partial_{T_{X,J}}^g A \right) \\
&= - 2 f_l \partial_p C_{k,\bar{l}} \zeta_p^* \otimes \zeta_k + \mathrm{Conj} \\
&= - \nabla_{g,J}^{1,0} A \nabla_g f .
\end{aligned}$$

On the other hand using the definition of  $\bar{\partial}_{T_{X,J}}^{*g,\Omega}$  and the expression (7.8) we obtain

$$\begin{aligned}
\bar{\partial}_{T_{X,J}}^{*g,\Omega} A &= - e^f \mathrm{Tr}_g \left[ \nabla_{g,J}^{1,0} (e^{-f} A) \right] \\
&= \bar{\partial}_{T_{X,J}}^{*g} A + \mathrm{Tr}_g (\partial_J f \otimes_J A) \\
&= \bar{\partial}_{T_{X,J}}^{*g} A + A \nabla_g f ,
\end{aligned}$$

and thus

$$\begin{aligned}
\partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g,\Omega} A &= \partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g} A + \partial_{T_{X,J}}^g (A \nabla_g f) \\
&= \partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g} A + \nabla_{g,J}^{1,0} A \nabla_g f + A \bar{\partial}_{T_{X,J}} \nabla_g f .
\end{aligned}$$

In fact

$$\begin{aligned}
2 \partial_{T_{X,J}}^g (A \nabla_g f) &= \nabla_g (A \nabla_g f) - J \nabla_{g,J} (A \nabla_g f) \\
&= \nabla_g A \nabla_g f + A \nabla_g^2 f \\
&\quad - J \nabla_{g,J} A \nabla_g f - J A \nabla_{g,J} \nabla_g f \\
&= 2 \nabla_{g,J}^{1,0} A \nabla_g f + A (\nabla_g^2 f + J \nabla_{g,J} \nabla_g f) \\
&= 2 \nabla_{g,J}^{1,0} A \nabla_g f + 2 A \bar{\partial}_{T_{X,J}} \nabla_g f .
\end{aligned}$$

We observe now that in degree 1 hold the identity

$$\partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g} + \frac{1}{2} \bar{\partial}_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g = J \left[ \partial_{T_{X,J}}^g, \left[ \omega^*, \partial_{T_{X,J}}^g \right] \right] = 0 . \quad (4.7)$$

In fact the first equality in (4.7) follows from a standard Kähler identity. The last equality in (4.7) follows from the graded Jacobi identity and the identity

$$\left( \partial_{T_{X,J}}^g \right)^2 = 0 .$$

Combining the previous formulas we infer the identity (4.4). Using the definition of the adjoint operator  $\bar{\partial}_{T_{X,J}}^{*,\Omega}$  and the expression (7.8) we expand the term

$$\begin{aligned}\bar{\partial}_{T_{X,J}}^{*,\Omega} \bar{\partial}_{T_{X,J}} B &= -2e^f \text{Tr}_g \left[ \nabla_{g,J}^{1,0} \left( e^{-f} \bar{\partial}_{T_{X,J}} B \right) \right] \\ &= \bar{\partial}_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} B + 2 \text{Tr}_g \left( \partial_J f \otimes_J \bar{\partial}_{T_{X,J}} B \right) \\ &= \bar{\partial}_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} B + 2 \nabla_g f \lrcorner \nabla_{g,J}^{0,1} B.\end{aligned}$$

In fact at the point  $p_0$  hold the following equalities

$$\begin{aligned}\text{Tr}_g \left( \partial_J f \otimes_J \bar{\partial}_{T_{X,J}} B \right) &= 2f_r \left( \bar{\zeta}_r \lrcorner_J \bar{\partial}_{T_{X,J}} B \right) \\ &= 2f_r \partial_{\bar{r}} B_{k,\bar{l}} \bar{\zeta}_k^* \otimes \zeta_l + \text{Conj} \\ &= \nabla_g f \lrcorner \nabla_{g,J}^{0,1} B.\end{aligned}$$

Then identity (4.5) follows combining lemma 4 with the identity identity

$$\xi \lrcorner \nabla_{g,J}^{1,0} B = \left( \xi \lrcorner \nabla_{g,J}^{0,1} B \right)_g^T. \quad (4.8)$$

This last follows immediately from the symmetry identity  $B = B_g^T$ . Using the definition of the operator  $\partial_{T_{X,J}}^{*,\Omega}$  and (7.7) we infer the equalities

$$\begin{aligned}\partial_{T_{X,J}}^{*,\Omega} \bar{\partial}_{T_{X,J}} B &= -2e^f \text{Tr}_g \left[ \nabla_{g,J}^{0,1} \left( e^{-f} \bar{\partial}_{T_{X,J}} B \right) \right] \\ &= \partial_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} B + 2 \text{Tr}_g \left( \bar{\partial}_J f \otimes_J \bar{\partial}_{T_{X,J}} B \right) \\ &= \partial_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} B - 2 \nabla_{g,J}^{0,1} B \nabla_g f.\end{aligned}$$

Indeed at the point  $p_0$  hold the equalities

$$\begin{aligned}\text{Tr}_g \left( \bar{\partial}_J f \otimes_J \bar{\partial}_{T_{X,J}} B \right) &= 2f_{\bar{k}} \left( \zeta_k \lrcorner_J \bar{\partial}_{T_{X,J}} B \right) \\ &= -2f_{\bar{k}} \partial_{\bar{p}} B_{k,\bar{l}} \bar{\zeta}_p^* \otimes \zeta_l + \text{Conj} \\ &= -\nabla_{g,J}^{0,1} B \nabla_g f.\end{aligned}$$

On the other hand combining the definition of  $\partial_{T_{X,J}}^{*g,\Omega}$  and (7.7) we obtain

$$\begin{aligned}\partial_{T_{X,J}}^{*g,\Omega} B &= -e^f \operatorname{Tr}_g \left[ \nabla_{g,J}^{0,1} (e^{-f} B) \right] \\ &= \partial_{T_{X,J}}^{*g} B + \operatorname{Tr}_g (\bar{\partial}_J f \otimes_J B) \\ &= \partial_{T_{X,J}}^{*g} B + B \nabla_g f ,\end{aligned}$$

and thus

$$\begin{aligned}\bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^{*g,\Omega} B &= \bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^{*g} B + \bar{\partial}_{T_{X,J}} (B \nabla_g f) \\ &= \bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^{*g} B + \nabla_{g,J}^{0,1} B \nabla_g f + B \bar{\partial}_{T_{X,J}} \nabla_g f .\end{aligned}$$

In fact

$$\begin{aligned}2 \bar{\partial}_{T_{X,J}} (B \nabla_g f) &= \nabla_g (B \nabla_g f) + J \nabla_{g,J} (B \nabla_g f) \\ &= \nabla_g B \nabla_g f + B \nabla_g^2 f \\ &\quad + J \nabla_{g,J} B \nabla_g f + J B \nabla_{g,J} \nabla_g f \\ &= 2 \nabla_{g,J}^{0,1} B \nabla_g f + B (\nabla_g^2 f + J \nabla_{g,J} \nabla_g f) \\ &= 2 \nabla_{g,J}^{0,1} B \nabla_g f + 2 B \bar{\partial}_{T_{X,J}} \nabla_g f .\end{aligned}$$

Then (4.6) follows combining the previous identities with the analogue of (4.7) in degree 1

$$\bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^{*g} + \frac{1}{2} \partial_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} = J \left[ \bar{\partial}_{T_{X,J}}, \left[ \omega^*, \bar{\partial}_{T_{X,J}} \right] \right] = 0 . \quad (4.9)$$

□

## 5 Variation formulas for the complex components of the $\Omega$ -Bakry-Emery-Ricci endomorphism

In this section and in the next one we will use in a crucial way the symmetry properties of the variations of Kähler structures obtained in [Pal1]. For this reason we will remind here a few key facts obtained in [Pal1]. First of all we

remind (see lemma 3, identity (16) in [Pal1]) that a smooth path  $(g_t, J_t)_t \subset \mathcal{KS}$  satisfies  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  iff the family  $(J_t)_{t \geq 0}$  is solution of the ODE

$$2 \dot{J}_t = J_t \dot{g}_t^* - \dot{g}_t^* J_t .$$

We remind also the following basic fact obtained in [Pal1].

**Lemma 6.** *Let  $(g_t)_{t \geq 0}$  be an arbitrary smooth family of Riemannian metrics and let  $(J_t)_{t \geq 0}$  be a family of endomorphisms of  $T_X$  solution of the ODE*

$$2 \dot{J}_t = J_t \dot{g}_t^* - \dot{g}_t^* J_t ,$$

*with initial conditions  $J_0^2 = -\mathbb{I}_{T_X}$  and  $(J_0)_{g_0}^T = -J_0$ . Then this conditions are preserved in time i.e.  $J_t^2 = -\mathbb{I}_{T_X}$  and  $(J_t)_{g_t}^T = -J_t$  for all  $t \geq 0$ .*

For any  $J \in \mathcal{J}$  and any  $J$ -invariant  $g \in \mathcal{M}$  we define in [Pal1] the vector space

$$\mathbb{D}_g^J := \left\{ v \in C^\infty(X, S_{\mathbb{R}}^2 T_X^*) \mid \nabla_{g,J}^{0,1} v_g^* \cdot \xi = \left( \nabla_{g,J}^{0,1} v_g^* \cdot \xi \right)_g^T, \forall \xi \in T_X \right\} .$$

With this notation hold the following important lemma obtained in [Pal1].

**Lemma 7.** *Let  $(g_t)_{t \geq 0}$  be an arbitrary smooth family of Riemannian metrics and let  $(J_t)_{t \geq 0}$  be a family of endomorphisms of  $T_X$  solution of the ODE*

$$2 \dot{J}_t = J_t \dot{g}_t^* - \dot{g}_t^* J_t ,$$

*with Kähler initial data  $(J_0, g_0)$ . Then  $(J_t, g_t)_{t \geq 0}$  is a smooth family of Kähler structures if and only if  $\dot{g}_t \in \mathbb{D}_{g_t}^{J_t}$  for all  $t \geq 0$ .*

Moreover (see [Pal1]) for any Kähler structure  $(J, g)$  hold the identity

$$\mathbb{D}_g^J = \left\{ v \in C^\infty(X, S_{\mathbb{R}}^2 T_X^*) \mid \partial_{T_X, J}^g (v_g^*)_J^{1,0} = 0, \bar{\partial}_{T_X, J} (v_g^*)_J^{0,1} = 0 \right\} . \quad (5.1)$$

We show now our general first variation formulas for the complex components of the  $\Omega$ -Bakry-Emery-Ricci endomorphism.

**Theorem 1.** *Let  $(g_t, J_t)_t \subset \mathcal{KS}$  be a smooth path such that  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$ . Let also  $f_t := \log \frac{dV_{g_t}}{\Omega}$ , with  $\Omega > 0$  a smooth volume form. Then hold the following first variation formulas for the complex components of the  $\Omega$ -Bakry-Emery-Ricci endomorphism;*

$$\begin{aligned} 2 \frac{d}{dt} \text{Ric}_{J_t}^*(\Omega)_{g_t} &= - \partial_{T_X, J_t}^{g_t} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \dot{g}_t^{0,1} - \left( \partial_{T_X, J_t}^{g_t} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \dot{g}_t^{0,1} \right)_t^T \\ &+ \left[ \text{Ric}_{J_t}^*(\Omega)_{g_t}, \dot{g}_t^{0,1} \right] - 2 \dot{g}_t^{1,0} \text{Ric}_{J_t}^*(\Omega)_{g_t} , \end{aligned} \quad (5.2)$$



$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \right) &= - \bar{\partial}_{T_X, J_t} \partial_{T_X, J_t}^{*g_t, \Omega} \dot{g}_t^{1,0} - \left( \bar{\partial}_{T_X, J_t} \partial_{T_X, J_t}^{*g_t, \Omega} \dot{g}_t^{1,0} \right)_t^T \\
&- (J_t \nabla_{g_t} f_t) \lrcorner (J_t \nabla_{g_t} \dot{g}_t^{0,1}) \\
&+ \left[ \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t, \dot{g}_t^* \right] \\
&- \dot{g}_t^{0,1} \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t - \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t \dot{g}_t^{0,1}, \quad (5.3)
\end{aligned}$$

with  $\dot{g}_t^{1,0} := (\dot{g}_t^*)_{J_t}^{1,0}$  and  $\dot{g}_t^{0,1} := (\dot{g}_t^*)_{J_t}^{0,1}$ .

*Proof.* For notation simplicity we set  $A_t := -\dot{g}_t^{0,1} = J_t \dot{J}_t$  and  $B_t := \dot{g}_t^{1,0}$ . Time deriving the trivial identity

$$\begin{aligned}
\text{Ric}_{J_t}^*(\Omega)_{g_t} &= -J_t \text{Ric}_{J_t}^*(\Omega)_{g_t} J_t \\
&= -J_t \omega_t^{-1} \text{Ric}_{J_t}(\Omega) J_t \\
&= -g_t^{-1} \text{Ric}_{J_t}(\Omega) J_t,
\end{aligned}$$

we obtain the expression

$$\begin{aligned}
2 \frac{d}{dt} \text{Ric}_{J_t}^*(\Omega)_{g_t} &= 2 \dot{g}_t^* g_t^{-1} \text{Ric}_{J_t}(\Omega) J_t - 2 g_t^{-1} \frac{d}{dt} (\text{Ric}_{J_t}(\Omega) J_t) \\
&= -2 \dot{g}_t^* \text{Ric}_{J_t}^*(\Omega)_{g_t} + (\text{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'')_t^* \\
&+ \nabla_{g_t} f_t \lrcorner \left( \nabla_{g_t, J_t}^{1,0} A_t - \nabla_{g_t, J_t}^{0,1} A_t \right) \\
&+ \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t A_t + A_t \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \\
&- (i \partial_{J_t} \bar{\partial}_{J_t} f_t)_t^* A_t - A_t (i \partial_{J_t} \bar{\partial}_{J_t} f_t)_t^*,
\end{aligned}$$

thanks to corollary 1 and thanks to the identity

$$\partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t = (i \partial_{J_t} \bar{\partial}_{J_t} f_t)_t^*,$$

which follows from the decomposition formula of the Hessian (2.2). Using lemmas 3, 7 and the equality (5.1) we expand the term

$$\begin{aligned}
(\text{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'')_t^* &= -\frac{1}{2} \nabla_{T_X, g_t}^{*\Omega} \partial_{T_X, J_t}^{g_t} A_t - \frac{1}{2} \left( \nabla_{T_X, g_t}^{*\Omega} \partial_{T_X, J_t}^{g_t} A_t \right)_t^T + \Delta_{g_t}^\Omega A_t \\
&= -\partial_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t + \Delta_{g_t} A_t + \nabla_{g_t} f_t \lrcorner \nabla_{g_t} A_t \\
&- \frac{1}{2} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t - \frac{1}{2} \left( \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t \right)_t^T,
\end{aligned}$$

thanks to formula (4.3). Using the identity (4.1) we obtain

$$\begin{aligned}
(\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'')^* &= -A_t \operatorname{Ric}_{g_t}^* - \operatorname{Ric}_{g_t}^* A_t \\
&- \nabla_{g_t} f_t \lrcorner \left( \nabla_{g_t, J_t}^{1,0} A_t - \nabla_{g_t, J_t}^{0,1} A_t \right) \\
&- \frac{1}{2} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t - \frac{1}{2} \left( \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t \right)_t^T,
\end{aligned}$$

Using formula (4.4) we deduce

$$\begin{aligned}
(\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t'')^* &= -A_t \operatorname{Ric}_{g_t}^* - \operatorname{Ric}_{g_t}^* A_t \\
&- \nabla_{g_t} f_t \lrcorner \left( \nabla_{g_t, J_t}^{1,0} A_t - \nabla_{g_t, J_t}^{0,1} A_t \right) \\
&+ \partial_{T_X, J_t}^{g_t} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} A_t + \left( \partial_{T_X, J_t}^{g_t} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} A_t \right)_t^T \\
&- A_t \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t - \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t A_t.
\end{aligned}$$

Then the variation formula (5.2) follows by plunging this expression in the previous formula for the variation of  $\operatorname{Ric}_{J_t}^*(\Omega)$  and rearranging the leading terms. We show now formula (5.3). By corollary 2 we infer the expression

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \right) &= -2 \dot{g}_t^* \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t + (\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t')^* \\
&- \nabla_{g_t} f_t \lrcorner \left( \nabla_{g_t, J_t}^{1,0} A_t - \nabla_{g_t, J_t}^{0,1} A_t \right) \\
&- \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t A_t - A_t \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \\
&+ \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t A_t + A_t \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t.
\end{aligned}$$

Using again lemmas 3, 7 and the equality (5.1) we expand the term

$$\begin{aligned}
(\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t')^* &= \frac{1}{2} \nabla_{T_X, g_t}^{*\Omega} \bar{\partial}_{T_X, J_t} B_t + \frac{1}{2} \left( \nabla_{T_X, g_t}^{*\Omega} \bar{\partial}_{T_X, J_t} B_t \right)_t^T - \Delta_{g_t}^\Omega B_t \\
&= \frac{1}{2} \partial_{T_X, J_t}^{*g_t, \Omega} \bar{\partial}_{T_X, J_t} B_t + \frac{1}{2} \left( \partial_{T_X, J_t}^{*g_t, \Omega} \bar{\partial}_{T_X, J_t} B_t \right)_t^T \\
&+ \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \bar{\partial}_{T_X, J_t} B_t - \nabla_{g_t} f_t \lrcorner \nabla_{g_t, J_t}^{0,1} B_t + \nabla_{g_t} f_t \lrcorner \nabla_{g_t, J_t}^{1,0} B_t \\
&- B_t \operatorname{Ric}_{g_t}^* + \operatorname{Ric}_{g_t}^* B_t - \Delta_{g_t} B_t - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} B_t,
\end{aligned}$$

thanks to the identity (4.5). Moreover simplifying and using the formula (4.2) we obtain the identity

$$(\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t)^* = \frac{1}{2} \partial_{T_X, J_t}^{*g_t, \Omega} \bar{\partial}_{T_X, J_t} B_t + \frac{1}{2} \left( \partial_{T_X, J_t}^{*g_t, \Omega} \bar{\partial}_{T_X, J_t} B_t \right)_t^T. \quad (5.4)$$

Applying the commutation formula (4.6) to the identity (5.4) we infer

$$\begin{aligned} (\operatorname{div}_{g_t}^\Omega \mathcal{D}_{g_t} \dot{g}_t)^* &= - \bar{\partial}_{T_X, J_t} \partial_{T_X, J_t}^{*g_t, \Omega} B_t - \left( \bar{\partial}_{T_X, J_t} \partial_{T_X, J_t}^{*g_t, \Omega} B_t \right)_t^T \\ &\quad + B_t \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t + \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t B_t. \end{aligned}$$

Then the variation formula (5.3) follows by plunging this expression in the previous expression of the variation of  $\bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t$ .  $\square$

We notice that one can deduce also the variation formula

$$\begin{aligned} 2 \frac{d}{dt} \operatorname{Ric}_{J_t}^* (\Omega)_{g_t} &= - \frac{1}{2} \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t - \frac{1}{2} \left( \bar{\partial}_{T_X, J_t}^{*g_t, \Omega} \partial_{T_X, J_t}^{g_t} A_t \right)_t^T \\ &\quad + A_t \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t + \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t A_t \\ &\quad - \left[ \operatorname{Ric}_{J_t}^* (\Omega)_{g_t}, A_t \right] - 2 B_t \operatorname{Ric}_{J_t}^* (\Omega)_{g_t}. \end{aligned}$$

## 6 Representation of the Soliton-Kähler-Ricci flow as a complex strictly parabolic system

In this section we will show that the Soliton-Kähler-Ricci flow (introduced in [Pal3]) generated by the Soliton-Ricci flow (introduced in [Pal2]) represents a strictly parabolic system of the complex components of the metric variation. Let

$$\mathbb{F}_g := \left\{ v \in C^\infty(X, S_{\mathbb{R}}^2 T_X^*) \mid \nabla_{T_X, g} v_g^* = 0 \right\}.$$

We remind (see [Pal2]) that the Soliton-Ricci flow is a strictly parabolic equation. Indeed we observe that the variation formula (2.1) implies directly that if  $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$  is a smooth family such that  $\dot{g}_t \in \mathbb{F}_{g_t}$  for all  $t \in \mathbb{R}$ , then

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t} (\Omega) = - \Delta_{g_t}^\Omega \dot{g}_t.$$

(see also [Pal2] for a different proof of this formula). We obtain in particular the variation formula

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t}^* (\Omega) = - \Delta_{g_t}^\Omega \dot{g}_t^* - 2 \dot{g}_t^* \operatorname{Ric}_{g_t}^* (\Omega). \quad (6.1)$$

In [Pal1] we show that for any Kähler structure  $(J, g)$  hold the identity

$$\mathbb{F}_g = \left\{ v \in \mathbb{D}_g^J \mid \bar{\partial}_{T_{X,J}}(v_g^*)_{J}^{1,0} = -\partial_{T_{X,J}}^g(v_g^*)_{J}^{0,1} \right\}. \quad (6.2)$$

It was also observed in [Pal1] that comparing the previous  $(1, 1)$ -forms by means of  $g$ -geodesic and  $J$ -holomorphic coordinates we can infer the identity

$$\mathbb{F}_g = \left\{ v \in \mathbb{D}_g^J \mid \xi \lrcorner \nabla_{g,J}^{0,1}(v_g^*)_{J}^{1,0} = \nabla_{g,J}^{1,0}(v_g^*)_{J}^{0,1} \cdot \xi, \quad \forall \xi \in T_X \right\}. \quad (6.3)$$

In order to see that the Soliton-Kähler-Ricci flow is a complex strictly parabolic system we need to obtain first variation formulas for the complex components of the  $\Omega$ -Bakry-Emery-Ricci endomorphism with respect to  $\mathbb{F}_g$ -valued variations of the metric. This formulas can be obtained with little effort from the general variation formulas (5.2), (5.3) by means of the identity (6.2). However we prefer to show this particular variation formulas in an independent way which avoids large part of the deep and heavy computations needed to proof formulas (5.2) and (5.3).

We start by showing the following quite elementary fact.

**Lemma 8.** *Let  $(J, g) \in \mathcal{KS}$  and let  $v \in \mathbb{F}_g$ . Then hold the commutation identities*

$$\partial_{T_{X,J}}^g \partial_{T_{X,J}}^{*g}(v_g^*)_{J}^{1,0} = \frac{1}{2} \bar{\partial}_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}}(v_g^*)_{J}^{1,0}, \quad (6.4)$$

$$\bar{\partial}_{T_{X,J}} \bar{\partial}_{T_{X,J}}^{*g}(v_g^*)_{J}^{0,1} = \frac{1}{2} \partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g(v_g^*)_{J}^{0,1}. \quad (6.5)$$

*Proof.* We observe first that using the identity (6.2) and the standard Kähler identities we infer

$$\begin{aligned} \partial_{T_{X,J}}^{*g}(v_g^*)_{J}^{1,0} &= J \left[ \omega^* \lrcorner \bar{\partial}_{T_{X,J}}(v_g^*)_{J}^{1,0} \right] \\ &= -J \left[ \omega^* \lrcorner \partial_{T_{X,J}}^g(v_g^*)_{J}^{0,1} \right] \\ &= \bar{\partial}_{T_{X,J}}^{*g}(v_g^*)_{J}^{0,1}. \end{aligned}$$

Thus

$$\begin{aligned} \partial_{T_{X,J}}^g \partial_{T_{X,J}}^{*g}(v_g^*)_{J}^{1,0} &= \partial_{T_{X,J}}^g \bar{\partial}_{T_{X,J}}^{*g}(v_g^*)_{J}^{0,1} \\ &= -\frac{1}{2} \bar{\partial}_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g(v_g^*)_{J}^{0,1} \\ &= \frac{1}{2} \bar{\partial}_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}}(v_g^*)_{J}^{1,0}, \end{aligned}$$

thanks to the identity (4.7). This concludes the proof of the identity (6.4).  
Moreover as before we infer the equalities

$$\begin{aligned}
\bar{\partial}_{T_{X,J}} \bar{\partial}_{T_{X,J}}^* (v_g^*)_{J}^{0,1} &= \bar{\partial}_{T_{X,J}} \partial_{T_{X,J}}^* (v_g^*)_{J}^{1,0} \\
&= -\frac{1}{2} \partial_{T_{X,J}}^* \bar{\partial}_{T_{X,J}} (v_g^*)_{J}^{1,0} \\
&= \frac{1}{2} \partial_{T_{X,J}}^* \partial_{T_{X,J}}^g (v_g^*)_{J}^{0,1},
\end{aligned}$$

thanks to the identity (4.9) This concludes the proof of the identity (6.5).  $\square$

We show now the following particular first variation formula for the  $J$ -anti-linear part of the Hessian.

**Lemma 9.** *Let  $(g_t, J_t)_t \subset \mathcal{KS}$  be a smooth path such that  $\dot{J}_t = (\dot{J}_t)_{g_t}^T$  and  $\dot{g}_t \in \mathbb{F}_{g_t}$ . Let also  $f_t := \log \frac{dV_{g_t}}{\Omega}$ , with  $\Omega > 0$  a smooth volume form. Then hold the following first variation identity for the  $J_t$ -anti-linear part of the endomorphism  $\nabla_{g_t}^2 f_t$*

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t \right) &= -2 \bar{\partial}_{T_{X,J_t}} \bar{\partial}_{T_{X,J_t}}^* \dot{g}_t^{0,1} - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} \\
&\quad - \dot{g}_t^{0,1} \partial_{T_{X,J_t}}^{g_t} \nabla_{g_t} f_t - \partial_{T_{X,J_t}}^{g_t} \nabla_{g_t} f_t \dot{g}_t^{0,1} \\
&\quad + \left[ \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t, \dot{g}_t^{0,1} \right] - 2 \dot{g}_t^{1,0} \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t. \quad (6.6)
\end{aligned}$$

with  $\dot{g}_t^{1,0} := (\dot{g}_t^*)_{J_t}^{1,0}$  and  $\dot{g}_t^{0,1} := (\dot{g}_t^*)_{J_t}^{0,1}$ .

*Proof.* Using the variation formula (3.1) we infer the identity

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t \right) &= -\nabla_{g_t} f_t \lrcorner J_t \nabla_{g_t} \dot{J}_t + \dot{J}_t \nabla_{g_t}^2 f_t J_t + J_t \nabla_{g_t}^2 f_t \dot{J}_t \\
&\quad + 2 \bar{\partial}_{T_{X,J_t}} \left( \frac{d}{dt} \nabla_{g_t} f_t \right).
\end{aligned}$$

Time deriving the definition of the gradient we obtain

$$d\dot{f}_t = \left( \frac{d}{dt} \nabla_{g_t} f_t \right) \lrcorner g_t + \nabla_{g_t} f_t \lrcorner \dot{g}_t,$$

thus

$$\frac{d}{dt} \nabla_{g_t} f_t = \nabla_{g_t} \dot{f}_t - \dot{g}_t^* \nabla_{g_t} f_t.$$

Combining this with the equalities  $2 \dot{f}_t = \text{Tr}_{g_t} \dot{g}_t$  and  $\dot{g}_t^{0,1} = -J_t \dot{J}_t$  we infer

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \right) &= \bar{\partial}_{T_X, J_t} \nabla_{g_t} \text{Tr}_{g_t} \dot{g}_t \\
&- 2 \bar{\partial}_{T_X, J_t} (\dot{g}_t^* \nabla_{g_t} f_t) + \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} \\
&- \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t \dot{g}_t^{0,1} + \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \dot{g}_t^{0,1} \\
&+ \dot{g}_t^{0,1} \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t - \dot{g}_t^{0,1} \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t .
\end{aligned}$$

Then

$$\begin{aligned}
2 \bar{\partial}_{T_X, J_t} (\dot{g}_t^* \nabla_{g_t} f_t) &= \nabla_{g_t} (\dot{g}_t^* \nabla_{g_t} f_t) + J_t \nabla_{g_t, J_t} (\dot{g}_t^* \nabla_{g_t} f_t) \\
&= \nabla_{g_t} \dot{g}_t^* \nabla_{g_t} f_t + \dot{g}_t^* \nabla_{g_t}^2 f_t \\
&+ J_t \nabla_{g_t, J_t} \dot{g}_t^* \nabla_{g_t} f_t + J_t \dot{g}_t^* \nabla_{g_t, J_t} \nabla_{g_t} f_t \\
&= 2 \nabla_{g, J}^{0,1} \dot{g}_t^* \nabla_{g_t} f_t + \dot{g}_t^{1,0} \nabla_{g_t}^2 f_t + \dot{g}_t^{0,1} \nabla_{g_t}^2 f_t \\
&+ \dot{g}_t^{1,0} J_t \nabla_{g_t, J_t} \nabla_{g_t} f_t - \dot{g}_t^{0,1} J_t \nabla_{g_t, J_t} \nabla_{g_t} f_t \\
&= 2 \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} \\
&+ 2 \dot{g}_t^{1,0} \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t + 2 \dot{g}_t^{0,1} \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t ,
\end{aligned}$$

since for any vector  $\xi \in T_X$  hold the identity

$$\nabla_{g, J}^{0,1} \dot{g}_t^* \cdot \xi = \xi \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} .$$

In fact decomposing and using the symmetries (6.3), (3.5) we infer

$$\begin{aligned}
\nabla_{g, J}^{0,1} \dot{g}_t^* \cdot \xi &= \nabla_{g, J}^{0,1} \dot{g}_t^{1,0} \cdot \xi + \nabla_{g, J}^{0,1} \dot{g}_t^{0,1} \cdot \xi \\
&= \xi \lrcorner \nabla_{g, J}^{1,0} \dot{g}_t^{0,1} + \xi \lrcorner \nabla_{g, J}^{0,1} \dot{g}_t^{0,1} \\
&= \xi \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} .
\end{aligned}$$

We remind now (see the proof of lemma 3 in [Pal2]) that the assumption  $\dot{g}_t \in \mathbb{F}_{g_t}$  implies the identity  $\nabla_{g_t} \text{Tr}_{g_t} \dot{g}_t = -\nabla_{g_t}^* \dot{g}_t^*$ . Combining the previous formulas

we infer the identity

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \right) &= - \bar{\partial}_{T_X, J_t} \nabla_{g_t}^* \dot{g}_t^* - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{g}_t^{0,1} \\
&+ \left[ \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t, \dot{g}_t^{0,1} \right] \\
&- \dot{g}_t^{0,1} \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t - \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t \dot{g}_t^{0,1} \\
&- 2 \dot{g}_t^{1,0} \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t .
\end{aligned}$$

The conclusion follows from the identities

$$\begin{aligned}
\bar{\partial}_{T_X, J_t} \nabla_{g_t}^* \dot{g}_t^* &= \bar{\partial}_{T_X, J_t} \partial_{T_X, J_t}^{*g_t} \dot{g}_t^{1,0} + \bar{\partial}_{T_X, J_t} \bar{\partial}_{T_X, J}^{*g} \dot{g}_t^{0,1} \\
&= 2 \bar{\partial}_{T_X, J_t} \bar{\partial}_{T_X, J_t}^{*g_t} \dot{g}_t^{0,1} .
\end{aligned}$$

□

**Corollary 3.** *Under the assumptions of lemma 9 hold the variation formula*

$$\begin{aligned}
2 \frac{d}{dt} \text{Ric}_{J_t}^* (\Omega)_{g_t} &= - 2 \partial_{T_X, J_t}^{g_t} \partial_{T_X, J_t}^{*g_t} \dot{g}_t^{1,0} - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} \dot{g}_t^{1,0} \\
&- \dot{g}_t^{0,1} \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t - \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \dot{g}_t^{0,1} - \left[ \text{Ric}_{g_t}^*, \dot{g}_t^{1,0} \right] \\
&+ \left[ \text{Ric}_{J_t}^* (\Omega)_{g_t}, \dot{g}_t^{0,1} \right] - 2 \dot{g}_t^{1,0} \text{Ric}_{J_t}^* (\Omega)_{g_t} . \tag{6.7}
\end{aligned}$$

*Proof.* Combining the commutation identity (6.5) with the formula (4.1) in the variation formula (5.3) we obtain

$$\begin{aligned}
2 \frac{d}{dt} \left( \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t \right) &= - \Delta_{g_t}^\Omega \dot{g}_t^{0,1} - \dot{g}_t^{0,1} \text{Ric}_{J_t}^* (\Omega)_{g_t} - \text{Ric}_{J_t}^* (\Omega)_{g_t} \dot{g}_t^{0,1} \\
&+ \left[ \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t, \dot{g}_t^{0,1} \right] - 2 \dot{g}_t^{1,0} \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t .
\end{aligned}$$

We remind in fact the identities  $\text{Ric}_{g_t}^* = \text{Ric}_{J_t}^* (\omega_t)_{g_t}$  and

$$\partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t = (i \partial_{J_t} \bar{\partial}_{J_t} f_t)_t^* .$$

(This last hold thanks to the decomposition formula of the Hessian (2.2).) On the other hand time deriving the complex decomposition formula (2.4) and using

the variation formula (6.1) we obtain

$$\begin{aligned}
2 \frac{d}{dt} \text{Ric}_{J_t}^*(\Omega)_{g_t} &= - \Delta_{g_t}^\Omega \dot{g}_t^* - 2 \dot{g}_t^* \text{Ric}_{g_t}^*(\Omega) - 2 \frac{d}{dt} \left( \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t \right) \\
&= - \Delta_{g_t}^\Omega \dot{g}_t^{1,0} + \left[ \text{Ric}_{J_t}^*(\Omega)_{g_t}, \dot{g}_t^{0,1} \right] \\
&\quad - \dot{g}_t^{0,1} \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t - \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t \dot{g}_t^{0,1} \\
&\quad - 2 \dot{g}_t^{1,0} \text{Ric}_{J_t}^*(\Omega)_{g_t} .
\end{aligned}$$

Then the conclusion follows combining formula (4.2) with the identity (6.4).  $\square$

We remind that with our conventions our adjoint operators

$$\nabla_{T_{X,g}}^*, \quad \partial_{T_{X,J}}^{*g}, \quad \bar{\partial}_{T_{X,J}}^{*g},$$

differ by the ones usually defined in the literature by a degree multiplicative factor. The motivation for our convention is to preserve functoriality between the Riemannian and Kähler geometry. We remind (see [Pal2]) that with our conventions the Hodge Laplacian operator acting on  $T_X$ -valued  $q$ -forms is defined as

$$\begin{aligned}
\Delta_{T_{X,g}} &:= \nabla_{T_{X,g}} \nabla_g^* + \nabla_g^* \nabla_{T_{X,g}} \\
&= \frac{1}{q} \nabla_{T_{X,g}} \nabla_{T_{X,g}}^* + \frac{1}{q+1} \nabla_{T_{X,g}}^* \nabla_{T_{X,g}} .
\end{aligned}$$

We define also the holomorphic and antiholomorphic Hodge Laplacian operators acting on  $T_X$ -valued  $q$ -forms as

$$\begin{aligned}
\Delta'_{T_{X,g,J}} &:= \frac{1}{q} \partial_{T_{X,J}}^g \partial_{T_{X,J}}^{*g} + \frac{1}{q+1} \partial_{T_{X,J}}^{*g} \partial_{T_{X,J}}^g , \\
\Delta''_{T_{X,g,J}} &:= \frac{1}{q} \bar{\partial}_{T_{X,J}} \bar{\partial}_{T_{X,J}}^{*g} + \frac{1}{q+1} \bar{\partial}_{T_{X,J}}^{*g} \bar{\partial}_{T_{X,J}} ,
\end{aligned}$$

with the usual convention  $\infty \cdot 0 = 0$ . This complex Hodge Laplacian operators coincide with the standard ones used in the literature. We remind that in the Kähler case hold the decomposition identity

$$\Delta_{T_{X,g}} = \Delta'_{T_{X,g,J}} + \Delta''_{T_{X,g,J}} .$$

Let  $(J_t, g_t)_{t \geq 0}$  be a solution of the  $\Omega$ -Soliton-Kähler-Ricci flow (see [Pal2]) generated by the  $\Omega$ -Soliton-Ricci flow (see [Pal2]). We set as usual  $A_t := -\dot{g}_t^{0,1} = J_t \dot{J}_t$  and  $B_t := \dot{g}_t^{1,0}$  and we observe the identity  $\dot{\omega}_t^* = B_t$ . Indeed time differentiating the equality  $\omega_t = g_t J_t$  we infer

$$\dot{\omega}_t = \dot{g}_t J_t + g_t \dot{J}_t = \dot{g}_t J_t + \omega_t \dot{g}_t^{0,1} .$$



Multiplying both sides with  $\omega^{-1} = -Jg^{-1}$  we obtain

$$\dot{\omega}_t^* = -J_t \dot{g}_t^* J_t + \dot{g}_t^{0,1} = \dot{g}_t^{1,0}.$$

With the previous notations the equation of the  $\Omega$ -Soliton-Kähler-Ricci flow generated by the  $\Omega$ -Soliton-Ricci flow rewrites as

$$\begin{cases} B_t = \text{Ric}_{J_t}^*(\Omega)_{g_t} - \mathbb{I}_{T_X}, \\ A_t = \bar{\partial}_{T_X, J_t} \nabla_{g_t} f_t, \\ \bar{\partial}_{T_X, J_t} B = \partial_{T_X, J_t} A. \end{cases} \quad (6.8)$$

The last equation in the system follows from the equality (6.2). Time differentiating the first two equations of the system (6.8) by means of the first variation formulas (6.7), (6.6) and using lemma 7 with the equality (5.1) we obtain the parabolic type evolution formulas

$$\begin{aligned} 2\dot{B}_t &= -2\Delta'_{T_X, g_t, J_t} B_t - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} B_t \\ &\quad - 2B_t^2 + 2A_t^2 + [B, A] - [\text{Ric}_{g_t}^*, B] - 2B, \end{aligned} \quad (6.9)$$

$$\begin{aligned} 2\dot{A}_t &= -2\Delta''_{T_X, g_t, J_t} A_t - \nabla_{g_t} f_t \lrcorner \nabla_{g_t} A_t \\ &\quad + A_t \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t + \partial_{T_X, J_t}^{g_t} \nabla_{g_t} f_t A_t - 2B_t A_t, \end{aligned} \quad (6.10)$$

which show the required conclusion.

## 7 Appendix

### 7.1 Canonical connections

Over any almost complex manifold  $(X, J)$  there exist a canonical connection of type  $(0, 1)$  over the vector bundle  $\Lambda_{\mathbb{C}}^p T_{X, J}^*$ ,

$$\bar{\partial}_{J, p} : C^\infty(\Lambda_{\mathbb{C}}^p T_{X, J}^*) \longrightarrow C^\infty(\Lambda_J^{0,1} T_X^* \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}^p T_{X, J}^*),$$

given by the formula

$$\bar{\partial}_{J, p} \alpha(\eta) := \eta_J^{0,1} \lrcorner \bar{\partial}_J \alpha,$$

for all  $\alpha \in C^\infty(X, \Lambda_{\mathbb{C}}^p T_{X, J}^*)$  and  $\eta \in C^\infty(X, T_X \otimes_{\mathbb{R}} \mathbb{C})$ . More explicitly

$$\begin{aligned} \bar{\partial}_{J, p} \alpha(\eta)(\xi_1, \dots, \xi_p) &= \eta^{0,1} \cdot \alpha(\xi_1, \dots, \xi_p) \\ &\quad + \sum_{l=1}^p (-1)^l \alpha([\eta^{0,1}, \xi_l^{1,0}]^{1,0}, \xi_1, \dots, \hat{\xi}_l, \dots, \xi_p), \end{aligned} \quad (7.1)$$

for all  $\xi_j \in C^\infty(X, T_X)$ . In order to show that  $\bar{\partial}_{J,p}$  is a connection of type  $(0, 1)$  we observe that for any  $f \in C^\infty(X, \mathbb{C})$  hold the identities

$$\begin{aligned}
\bar{\partial}_{J,p}(f\alpha)(\eta) &= \eta^{0,1} \lrcorner \bar{\partial}_J(f\alpha) \\
&= \eta^{0,1} \lrcorner (\bar{\partial}_J f \wedge \alpha + f \bar{\partial}_J \alpha) \\
&= \bar{\partial}_J f(\eta) \cdot \alpha - \bar{\partial}_J f \wedge (\eta^{0,1} \lrcorner \alpha) + f \bar{\partial}_J \alpha(\eta) \\
&= (\bar{\partial}_J f \otimes \alpha + f \bar{\partial}_J \alpha)(\eta),
\end{aligned}$$

since  $\alpha$  is of type  $(p, 0)$ . In a similar way there exists a canonical connection of type  $(1, 0)$  over the vector bundle  $\Lambda_{\mathbb{C}}^p T_{X,-J}^*$ ,

$$\partial_{J,p} : C^\infty(\Lambda_{\mathbb{C}}^p T_{X,-J}^*) \longrightarrow C^\infty(\Lambda_J^{1,0} T_X^* \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}^p T_{X,-J}^*),$$

given by the formula

$$\partial_{J,p} \alpha(\eta) := \eta_J^{1,0} \lrcorner \partial_J \alpha,$$

for all  $\alpha \in C^\infty(X, \Lambda_{\mathbb{C}}^p T_{X,-J}^*)$  and  $\eta \in C^\infty(X, T_X \otimes_{\mathbb{R}} \mathbb{C})$ . More explicitly

$$\begin{aligned}
\partial_{J,p} \alpha(\eta)(\xi_1, \dots, \xi_p) &= \eta^{1,0} \cdot \alpha(\xi_1, \dots, \xi_p) \\
&+ \sum_{l=1}^p (-1)^l \alpha([\eta^{1,0}, \xi_l^{0,1}]^{0,1}, \xi_1, \dots, \hat{\xi}_l, \dots, \xi_p), \quad (7.2)
\end{aligned}$$

for all  $\xi_j \in C^\infty(X, T_X)$ . We observe in particular the identity  $\bar{\partial}_{J,p} \alpha = \overline{\partial_{J,p} \bar{\alpha}}$ . For  $p = 1$  the connection  $\bar{\partial}_{J,1}$  writes as

$$\bar{\partial}_{J,1} \alpha(\eta) \cdot \xi = \eta \cdot \alpha(\xi) - \alpha([\eta, \xi]^{1,0}),$$

for all  $\eta \in C^\infty(X, T_{X,J}^{0,1})$  and  $\xi \in C^\infty(X, T_{X,J}^{1,0})$ . We infer that its dual connection

$$\bar{\partial}_{T_{X,J}^{1,0}} : C^\infty(T_{X,J}^{1,0}) \longrightarrow C^\infty(\Lambda_J^{0,1} T_X^* \otimes_{\mathbb{C}} T_{X,J}^{1,0}),$$

over the bundle  $T_{X,J}^{1,0}$ , which is defined by the formula

$$(\bar{\partial}_{J,1} \alpha) \cdot \xi = \bar{\partial}_J(\alpha \cdot \xi) - \alpha \cdot \bar{\partial}_{T_{X,J}^{1,0}} \xi,$$

satisfies the identity

$$\bar{\partial}_{T_{X,J}^{1,0}} \xi(\eta) = [\eta, \xi]^{1,0}.$$

Moreover the canonical  $\mathbb{C}$ -isomorphism between  $T_{X,J}^{1,0}$  and  $T_{X,J}$  induces a canonical connection

$$\bar{\partial}_{T_{X,J}} : C^\infty(T_{X,J}) \longrightarrow C^\infty(T_{X,-J}^* \otimes_{\mathbb{C}} T_{X,J}),$$

of type  $(0, 1)$  over the bundle  $T_{X,J}$ . Explicitly for any  $\xi, \eta \in C^\infty(X, T_X)$ ,

$$\begin{aligned}\bar{\partial}_{T_{X,J}} \xi(\eta) &:= \bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0}(\eta) + \overline{\bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0}(\eta)} \\ &= [\eta^{0,1} \xi^{1,0}]^{1,0} + [\eta^{1,0}, \xi^{0,1}]^{0,1}.\end{aligned}$$

Moreover the connection  $\bar{\partial}_{T_{X,J}}$  induces in a natural way a connection

$$\partial_{T_{X,-J}} : C^\infty(T_{X,-J}) \longrightarrow C^\infty(T_{X,J}^* \otimes_{\mathbb{C}} T_{X,-J}),$$

of type  $(1, 0)$  over the bundle  $T_{X,-J}$  by the formula

$$\partial_{T_{X,-J}} \xi(\eta) := \bar{\partial}_{T_{X,J}} \xi(\bar{\eta}),$$

for all  $\xi \in C^\infty(X, T_X)$  and  $\eta \in T_X \otimes_{\mathbb{R}} \mathbb{C}$ . On the other hand the identity (7.2) writes for  $p = 1$  as

$$\partial_{J,1} \alpha(\eta) \cdot \xi = \eta \cdot \alpha(\xi) - \alpha([\eta, \xi]^{0,1}),$$

for all  $\eta \in C^\infty(X, T_{X,J}^{1,0})$  and  $\xi \in C^\infty(X, T_{X,J}^{0,1})$ . We infer that its dual connection

$$\partial_{T_{X,J}^{0,1}} : C^\infty(T_{X,J}^{0,1}) \longrightarrow C^\infty(\Lambda_J^{1,0} T_X^* \otimes_{\mathbb{C}} T_{X,J}^{0,1}),$$

over the bundle  $T_{X,J}^{0,1}$ , which is defined by the formula

$$(\partial_{J,1} \alpha) \cdot \xi = \partial_J(\alpha \cdot \xi) - \alpha \cdot \partial_{T_{X,J}^{0,1}} \xi,$$

satisfies the identity

$$\partial_{T_{X,J}^{0,1}} \xi(\eta) = [\eta, \xi]^{0,1}.$$

We deduce that for all  $\xi \in C^\infty(X, T_X)$  and  $\eta \in T_X \otimes_{\mathbb{R}} \mathbb{C}$  hold the identity

$$\begin{aligned}\bar{\partial}_{T_{X,J}} \xi(\eta) &= \bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0}(\eta) + \overline{\bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0}(\eta)} \\ &= \bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0}(\eta) + \partial_{T_{X,J}^{0,1}} \xi^{0,1}(\bar{\eta}).\end{aligned}$$

We conclude that the connection  $\partial_{T_{X,-J}}$  is the dual of the connection  $\partial_{J,1}$  over the bundle  $T_{X,-J}$ .

## 7.2 The Levi-Civita connection of a Kähler metric

For convenience the notation for the  $\partial$  operator in this subsection is slightly different from the one used in the paper. Let  $(X, J, \omega)$  be an almost hermitian manifold. Let  $g := \omega(\cdot, J\cdot)$  be the induced Riemannian metric and  $h := g - i\omega$

be the induced hermitian metric over  $T_{X,J}$ . The hermitian data determines two connections of type  $(1,0)$ ;

$$\partial_{T_{X,J}}^\omega := h^{-1} \cdot \partial_{J,1} \cdot h : C^\infty(T_{X,J}) \longrightarrow C^\infty(\Lambda_J^{1,0} T_X^* \otimes_{\mathbb{C}} T_{X,J}) ,$$

and

$$\partial_{T_{X,J}^{1,0}}^\omega := \omega^{-1} \cdot \partial_{J,1} \cdot \omega : C^\infty(T_{X,J}^{1,0}) \longrightarrow C^\infty(\Lambda_J^{1,0} T_X^* \otimes_{\mathbb{C}} T_{X,J}^{1,0}) ,$$

where  $h$  and  $\omega$  are considered as morphisms  $h : T_{X,J} \rightarrow T_{X,-J}^*$  and  $\omega : T_X \rightarrow T_X^*$ . The trivial identities

$$\begin{aligned} h(\xi, \eta) &= h(\xi^{1,0}, \eta) \\ &= h(\xi^{1,0}, \eta^{0,1}) \\ &= -2i\omega(\xi^{1,0}, \eta^{0,1}) \\ &= -2i\omega(\xi^{1,0}, \eta) , \end{aligned}$$

for all  $\xi, \eta \in C^\infty(X, T_X)$  imply

$$\xi \lrcorner h = -2i\xi^{1,0} \lrcorner \omega . \quad (7.3)$$

This combined with the definition of  $\partial_{T_{X,J}}^\omega$  gives

$$\begin{aligned} \mu &:= \partial_{T_{X,J}}^\omega \xi(\eta) := h^{-1} [\eta \lrcorner \partial_{J,1} (\xi \lrcorner h)] \\ &= h^{-1} [\eta^{1,0} \lrcorner \partial_J (\xi \lrcorner h)] \\ &= -2h^{-1} [\eta^{1,0} \lrcorner i\partial_J (\xi^{1,0} \lrcorner \omega)] . \end{aligned}$$

If we set

$$\alpha := -2[\eta^{1,0} \lrcorner i\partial_J (\xi^{1,0} \lrcorner \omega)] = \mu \lrcorner h ,$$

then the identity (7.3) applied to  $\mu$  implies

$$\mu^{1,0} = \frac{i}{2} \omega^{-1} \alpha = \omega^{-1} [\eta^{1,0} \lrcorner \partial_J (\xi^{1,0} \lrcorner \omega)] .$$

Moreover by definition

$$\begin{aligned} \partial_{T_{X,J}^{1,0}}^\omega \xi^{1,0}(\eta) &:= \omega^{-1} [\eta \lrcorner \partial_{J,1} (\xi^{1,0} \lrcorner \omega)] \\ &= \omega^{-1} [\eta^{1,0} \lrcorner \partial_J (\xi^{1,0} \lrcorner \omega)] \\ &= -Jg^{-1} [\eta^{1,0} \lrcorner i\partial_J (\xi^{1,0} \lrcorner g)] . \end{aligned}$$

We infer

$$\partial_{T_{X,J}}^\omega \xi(\eta) = \partial_{T_{X,J}^{1,0}}^\omega \xi^{1,0}(\eta) + \overline{\partial_{T_{X,J}^{1,0}}^\omega \xi^{1,0}(\eta)}.$$

In conclusion we obtain the formula

$$\partial_{T_{X,J}}^\omega \xi(\eta) = -Jg^{-1}[\eta^{1,0} \lrcorner i\partial_J(\xi^{1,0} \lrcorner g) - \eta^{0,1} \lrcorner i\bar{\partial}_J(\xi^{0,1} \lrcorner g)]. \quad (7.4)$$

The Chern connection

$$D_{T_{X,J}}^\omega := \partial_{T_{X,J}}^\omega + \bar{\partial}_{T_{X,J}},$$

is  $h$ -hermitian. From now on we assume that  $(X, J, \omega)$  is a Kähler manifold, i.e.  $\nabla_g J = 0$ . In this case the Levi-Civita connection  $\nabla_g$  coincides with the Chern connection  $D_{T_{X,J}}^\omega$ . We consider also the components

$$\begin{aligned} \nabla_{g,J}^{1,0} \xi &:= \frac{1}{2} (\nabla_g \xi - J \nabla_g \xi \cdot J), \\ \nabla_{g,J}^{0,1} \xi &:= \frac{1}{2} (\nabla_g \xi + J \nabla_g \xi \cdot J), \end{aligned}$$

of the complexified Levi-Civita connection. The identities

$$\begin{aligned} \nabla_{g,J}^{1,0} &= \partial_{T_{X,J}}^\omega, \\ \nabla_{g,J}^{0,1} &= \bar{\partial}_{T_{X,J}}, \end{aligned}$$

hold only in restriction to the real tangent bundle  $T_X$ . In general hold the identities

$$\begin{aligned} \nabla_{g,J}^{1,0} \xi &= \partial_{T_{X,J}^{1,0}}^\omega \xi^{1,0} + \overline{\partial_{T_{X,J}^{1,0}}^\omega \xi^{0,1}}, \\ \nabla_{g,J}^{0,1} \xi &= \bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0} + \overline{\bar{\partial}_{T_{X,J}^{1,0}} \xi^{0,1}} \\ &= \bar{\partial}_{T_{X,J}^{1,0}} \xi^{1,0} + \partial_{T_{X,J}^{0,1}} \xi^{0,1}. \end{aligned}$$

Indeed  $\nabla_{g,J}^{1,0}$  and  $\nabla_{g,J}^{0,1}$  are respectively the  $\mathbb{C}$ -linear extensions of  $\partial_{T_{X,J}}^\omega$  and  $\bar{\partial}_{T_{X,J}}$  to the complexified tangent bundle  $T_X \otimes_{\mathbb{R}} \mathbb{C}$ . In particular  $\nabla_{g,J}^{0,1}$  is independent of the metric  $g$ . Let now  $(\zeta_k)_{k=1}^n \subset \mathcal{O}(U, T_{X,J}^{1,0})$  be a local holomorphic frame and write  $\xi = \xi'_k \zeta_k + \xi''_k \bar{\zeta}_k$ . Then hold the local expressions

$$\begin{aligned} \nabla_{g,J}^{1,0} \xi &= (\zeta_p \cdot \xi'_k + A_{k,l}^p \xi'_l) \zeta_p^* \otimes \zeta_k + (\bar{\zeta}_p \cdot \xi''_k + \overline{A_{k,l}^p} \xi''_l) \bar{\zeta}_p^* \otimes \bar{\zeta}_k, \\ \nabla_{g,J}^{0,1} \xi &= (\bar{\zeta}_p \cdot \xi'_k) \bar{\zeta}_p^* \otimes \zeta_k + (\zeta_p \cdot \xi''_k) \zeta_p^* \otimes \bar{\zeta}_k, \end{aligned}$$

where  $A_{k,l}^p := (\zeta_p \cdot \omega_{l,\bar{r}}) \omega^{r,\bar{k}}$  and

$$\omega = \frac{i}{2} \omega_{k,\bar{l}} \zeta_k^* \wedge \bar{\zeta}_l^* .$$

In particular we find the following expressions;

$$\nabla_g \zeta_l = A_{k,l}^p \zeta_p^* \otimes \zeta_k , \quad \nabla_g \bar{\zeta}_l = \overline{A_{k,l}^p} \bar{\zeta}_p^* \otimes \bar{\zeta}_k . \quad (7.5)$$

We infer the following identities for the dual connection on the complexified cotangent bundle  $T_x^* \otimes_{\mathbb{R}} \mathbb{C}$ ;

$$\nabla_g \zeta_l^* = - A_{l,k}^p \zeta_p^* \otimes \zeta_k^* , \quad \nabla_g \bar{\zeta}_l^* = - \overline{A_{l,k}^p} \bar{\zeta}_p^* \otimes \bar{\zeta}_k^* . \quad (7.6)$$

### 7.3 Complex operators acting on alternating tensors

Let  $(F, h)$  be a hermitian vector bundle over a Kähler manifold  $(M, J, g)$  (of complex dimension  $n$ ) equipped with a  $h$ -hermitian connection  $\nabla_F$  and let  $\nabla$  be the induced hermitian connection over the hermitian vector bundle

$$((T_M^*)^{\otimes p} \otimes_{\mathbb{R}} F, \langle \cdot, \cdot \rangle) ,$$

where  $\langle \cdot, \cdot \rangle$  is the induced hermitian product. We define the complex operators

$$2 \partial_F := \nabla_F - J_F \nabla_{F,J} ,$$

$$2 \bar{\partial}_F := \nabla_F + J_F \nabla_{F,J} ,$$

acting on sections of  $F$ . We will denote with the same notations their extension to the sheaf of  $F$ -valued differential forms. We notice the decomposition formula  $\nabla_F = \partial_F + \bar{\partial}_F$ . We define also the complex operators

$$2 \nabla_J^{1,0} := \nabla - J_F \nabla_{J\cdot} ,$$

$$2 \nabla_J^{0,1} := \nabla + J_F \nabla_{J\cdot} ,$$

acting on  $F$ -valued tensors. Let now  $(e_k)_k$  be a  $J$ -complex and  $gJ$ -orthonormal basis of  $T_M$ . We expand the formula (see the appendix in [Pal1])

$$\begin{aligned} \nabla_F^* \alpha &= - p \operatorname{Tr}_g \nabla \alpha \\ &= - p \nabla \alpha (e_j, e_j, \cdot) - p \nabla \alpha (J e_j, J e_j, \cdot) \\ &= - 2 p \nabla \alpha (\zeta_j, \bar{\zeta}_j, \cdot) - 2 p \nabla \alpha (\bar{\zeta}_j, \zeta_j, \cdot) . \end{aligned}$$

The fact that the metric  $g$  is Kähler implies that the operator  $\nabla_{\bullet}\alpha$  preserves the be-degrees. Thus by be-degree reasons we infer the formulas

$$\partial_F^* \alpha = -2p \nabla \alpha(\bar{\zeta}_j, \zeta_j, \cdot) = -p \operatorname{Tr}_g \nabla_J^{0,1} \alpha, \quad (7.7)$$

$$\bar{\partial}_F^* \alpha = -2p \nabla \alpha(\zeta_j, \bar{\zeta}_j, \cdot) = -p \operatorname{Tr}_g \nabla_J^{1,0} \alpha. \quad (7.8)$$

We observe also the formulas

$$\partial_F \alpha(\xi_0, \dots, \xi_p) = \sum_{j=0}^p (-1)^j \nabla_J^{1,0} \alpha(\xi_j, \xi_0, \dots, \hat{\xi}_j, \dots, \xi_p), \quad (7.9)$$

$$\bar{\partial}_F \alpha(\xi_0, \dots, \xi_p) = \sum_{j=0}^p (-1)^j \nabla_J^{0,1} \alpha(\xi_j, \xi_0, \dots, \hat{\xi}_j, \dots, \xi_p). \quad (7.10)$$

It is sufficient to show the identities (7.9), (7.10) for a real basis  $(\xi_j)$  of  $T_M$ . So let  $(z_1, \dots, z_n)$  be  $J$ -holomorphic and  $g$ -geodesic coordinates centered at a point  $p_0$  and let  $\xi_j := \frac{\partial}{\partial x_j}$  or  $\xi_j := \frac{\partial}{\partial y_j}$ , where  $z_j = x_j + i y_j$ . We infer

$$\begin{aligned} \partial_F \alpha(\xi_0, \dots, \xi_p) &= \sum_{j=0}^p (-1)^j \partial_{F, \xi_j} [\alpha(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_p)] \\ &= \frac{1}{2} \sum_{j=0}^p (-1)^j (\nabla_{F, \xi_j} - J_F \nabla_{F, J\xi_j}) [\alpha(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_p)], \end{aligned}$$

and thus at the point  $p_0$  hold the equalities

$$\begin{aligned} \partial_F \alpha(\xi_0, \dots, \xi_p) &= \frac{1}{2} \sum_{j=0}^p (-1)^j (\nabla_{\xi_j} \alpha - J_F \nabla_{J\xi_j} \alpha)(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_p) \\ &= \sum_{j=0}^p (-1)^j \nabla_J^{1,0} \alpha(\xi_j, \xi_0, \dots, \hat{\xi}_j, \dots, \xi_p), \end{aligned}$$

since  $\nabla_g \xi_j(p_0) = 0$ . The proof of the identity (7.10) is quite similar.

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